

SIMPLE ENDOTRIVIAL MODULES FOR LINEAR, UNITARY AND EXCEPTIONAL GROUPS

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ABSTRACT. Motivated by a recent result of Robinson showing that simple endotrivial modules essentially come from quasi-simple groups we classify such modules for finite special linear and unitary groups as well as for exceptional groups of Lie type. Our main tool is a lifting result for endotrivial modules obtained in a previous paper which allows us to apply character theoretic methods. As one application we prove that the ℓ -rank of quasi-simple groups possessing a faithful simple endotrivial module is at most 2. As a second application we complete the proof that principal blocks of finite simple groups cannot have Loewy length 4, thus answering a question of Koshitani, Külshammer and Sambale. Our results also imply a vanishing result for irreducible characters of special linear and unitary groups.

1. INTRODUCTION

Let G be a finite group and k a field of prime characteristic ℓ dividing $|G|$. A kG -module V is called *endotrivial* if $V \otimes V^* \cong k \oplus P$, with a projective kG -module P . Endotrivial modules have seen a considerable interest in the last fifteen years. Due to a recent result of Robinson, the focus has moved to simple endotrivial modules for quasi-simple groups. In our predecessor paper [16] we classified such modules for several families of finite quasi-simple groups. The present paper is a continuation of this project. We obtain a complete classification for special linear and unitary groups and almost complete results for groups of exceptional Lie type.

We also give a necessary and sufficient condition for a trivial source module to be endotrivial, depending only on the values of the associated ordinary character on ℓ -elements of the group. This allows us to settle some cases left open in [16] for simple modules of sporadic groups.

It turned out in our previous work [16] that simple modules of quasi-simple groups G are rarely ever endotrivial, and if such modules exist at all, then this seems to severely restrict the structure of Sylow ℓ -subgroups of G . Combining our new results with those of [16] and [17] we can show Conjecture 1.1 in [16]:

Theorem 1.1. *Let G be a finite quasi-simple group having a faithful simple endotrivial module in characteristic ℓ . Then the Sylow ℓ -subgroups of G have rank at most 2.*

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In the case of cyclic Sylow ℓ -subgroups, we calculated in [16, §3] the number of ℓ -blocks of a group containing simple endotrivial modules and proved that they correspond to non-exceptional liftable characters. In ℓ -rank 2, simple endotrivial modules only seldom occur. We show that the following covering groups of linear, unitary and exceptional groups do have faithful simple endotrivial modules in ℓ -rank 2: for $\ell = 3$, the groups $2.L_3(4)$, $4_1.L_3(4)$, $4_2.L_3(4)$ and $U_3(q)$ with $q \equiv 2, 5 \pmod{9}$, for $\ell = 5$, the groups $F_4(2)$, $2.F_4(2)$, and for $\ell = 7$, the group $2.F_4(2)$. We note that all these modules have trivial source, apart from the ones for $4_1.L_3(4)$.

As a further application of our results we are able to complete an investigation begun by Koshitani, Külshammer and Sambale [12] and show:

Theorem 1.2. *Let G be a finite non-abelian simple group and $\ell > 2$ a prime such that Sylow ℓ -subgroups of G are noncyclic. Then the principal ℓ -block of G cannot have Loewy length 4.*

Our proofs also yield a vanishing result for characters of linear and unitary groups of large enough ℓ -rank:

Theorem 1.3. *Let $\ell > 2$ be a prime and G a finite quasi-simple covering group of $L_n(q)$ or $U_n(q)$ of ℓ -rank at least 3. Then for any non-trivial $\chi \in \text{Irr}(G)$ there exists an ℓ -singular element $g \in G$ with $\chi(g) = 0$, unless one of:*

- (1) $\ell|q$ is the defining characteristic of G ;
- (2) $\ell = 5$, $G = L_5(q)$ with $5|(q-1)$ and $\chi(1) = q^2\Phi_5$; or
- (3) $\ell = 5$, $G = U_5(q)$ with $5|(q+1)$ and $\chi(1) = q^2\Phi_{10}$.

One of the main tools for our investigations is our earlier result [16, Thm. 1.3] which asserts that all endotrivial modules are liftable to characteristic 0. This allows us to apply character theoretic methods in our investigation. More specifically, we can employ the ordinary character theory of groups of Lie type as developed by Lusztig.

The paper is built up as follows. In Section 2 we derive Theorem 2.2 and apply it first in Corollary 2.4 to rule out some examples in sporadic and exceptional type groups, and in Corollary 2.9 to compute the torsion subgroup $TT(G)$ of $T(G)$ for several small sporadic simple groups. In Section 3 we show that special linear groups in rank at least 3 have no faithful simple endotrivial modules (see Theorem 3.10). In Section 4 we classify simple endotrivial modules for special unitary groups (see Theorem 4.5) and complete the proof of Theorem 1.1 (in Section 4.3) and of Theorem 1.3 (in Section 4.5). We investigate simple endotrivial modules for exceptional type groups in Section 5, leaving open only one situation in groups of types E_6 , 2E_6 and E_7 , respectively, see Theorem 5.2. Finally, in Section 6 we complete the proof of Theorem 1.2.

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2. CHARACTERS OF TRIVIAL SOURCE ENDOTRIVIAL MODULES

Let G be a finite group, ℓ a prime number dividing $|G|$, k an algebraically closed field of characteristic ℓ . Let (K, \mathcal{O}, k) be a splitting ℓ -modular system for G and its subgroups. Let $1_G \in \text{Irr}(G)$ denote the trivial character of G .

If M is an indecomposable trivial source kG -module, then M lifts uniquely to an indecomposable trivial source $\mathcal{O}G$ -lattice, denoted henceforth by \hat{M} . Let $\chi_{\hat{M}}$ denote the ordinary character of G afforded by \hat{M} . The following was proven by Landrock–Scott, see [15, II, Lem. 12.6]:

Lemma 2.1. *Let M be an indecomposable trivial source kG -module and $x \in G$ be an ℓ -element. Then:*

- (a) $\chi_{\hat{M}}(x) \geq 0$ is an integer (corresponding to the multiplicity of the trivial $k\langle x \rangle$ -module as a direct summand of $M|_{\langle x \rangle}$);
- (b) $\chi_{\hat{M}}(x) \neq 0$ if and only if x belongs to a vertex of M .

From this we derive the following necessary and sufficient condition:

Theorem 2.2. *Let M be an indecomposable trivial source kG -module. Then M is endotrivial if and only if $\chi_{\hat{M}}(x) = 1$ for all non-trivial ℓ -elements $x \in G$.*

Proof. If $\dim_k M = 1$, then M is endotrivial and satisfies $\chi_{\hat{M}}(x) = 1$ for all ℓ -elements $x \in G$, therefore we may assume $\dim_k M > 1$. We may also assume $\dim_k M$ is prime to ℓ . Indeed, on the one hand endotrivial modules have dimension prime to ℓ (see [16, Lem. 2.1]), and on the other hand if the sufficient condition is assumed, then $\dim_k M \equiv \chi_{\hat{M}}(x) = 1 \pmod{\ell}$ for any non-trivial ℓ -element $x \in G$. Thus, by [2, Thm. 2.1], the trivial module occurs as a direct summand of $M \otimes_k M^*$ with multiplicity 1. Since M has trivial source we may write

$$M \otimes_k M^* \cong k \oplus N_1 \oplus \dots \oplus N_r$$

where N_1, \dots, N_r are non-trivial indecomposable trivial source modules. Clearly M is endotrivial if and only if the module N_i has vertex $\{1\}$ for every $1 \leq i \leq r$. At the level of characters we have

$$\chi_{\hat{M}}\chi_{\hat{M}^*} = 1_G + \chi_{\hat{N}_1} + \dots + \chi_{\hat{N}_r}.$$

Thus, by Lemma 2.1(b), M is endotrivial if and only if $(\chi_{\hat{M}}\chi_{\hat{M}^*})(x) = 1$ for every non-trivial ℓ -element $x \in G$. Moreover since $\dim_k M$ is prime to ℓ , M has vertex a Sylow ℓ -subgroup P , so that, by Lemma 2.1(a) and (b), $\chi_{\hat{M}}(x)$ is a positive integer for every non-trivial ℓ -element $x \in G$. The claim follows. \square

Henceforth we denote by $T(G)$ the abelian group of isomorphism classes of indecomposable endotrivial modules, with multiplication induced by the tensor product \otimes_k . Then $T(G)$ is known to be finitely generated. For notation and background material on $T(G)$ we refer to the survey [4] and the references therein.

Under the assumption that the normal ℓ -rank of the group G is greater than one, trivial source endotrivial modules coincide with torsion endotrivial modules. Then Theorem 2.2 says that the torsion subgroup $TT(G)$ of $T(G)$ is a function of the character table of G .

Corollary 2.3. *Assume the Sylow ℓ -subgroups of G are neither cyclic, nor semi-dihedral, nor generalised quaternion. If V is a self-dual endotrivial kG -module, then $\chi_{\hat{V}}(1) \equiv 1 \pmod{|G|_{\ell}}$ and $\chi_{\hat{V}}(x) = 1$ for all non-trivial ℓ -elements $x \in G$.*

Proof. If V is self-dual, then the class of V is a torsion element of the group of endotrivial modules $T(G)$. If $P \in \text{Syl}_{\ell}(G)$, then restriction to P induces a group homomorphism

$\text{Res}_P^G : T(G) \longrightarrow T(P) : [M] \mapsto [M|_P]$, where under the assumptions on P the group $T(P)$ is torsion-free (see [23, §7 and §8]). It follows that $V|_P \cong k \oplus (\text{proj})$, that is, V is a trivial source kG -module. The congruence modulo $|G|_\ell$ is immediate and the second claim is given by Theorem 2.2. \square

This allows us to settle some of the cases left open in [16] corresponding to self-dual modules not satisfying the conclusion of Corollary 2.3:

Corollary 2.4. *The candidate characters for the sporadic groups Fi'_{24} and M from [16, Table 6] are not endotrivial. Similarly neither are the unipotent characters $\phi_{80,7}$ for $E_6(q)$ nor $\phi_{16,5}$ for ${}^2E_6(q)$ from [16, Table 4].*

The following examples show how Theorem 2.2 enables us to find torsion endotrivial modules using induction and restriction of characters and to settle some other cases for covering groups of sporadic simple groups left open in [16, Table 6] or which could so far be discarded via **Magma** computations only. Below we denote ordinary irreducible characters by their degrees, and the labelling of characters and blocks is that of the **GAP** character table library [22].

Example 2.5 (The character of degree 55 of M_{22} in characteristic $\ell = 3$).

Let $G = M_{22}$ and let $P \in \text{Syl}_3(G)$. Let V_{55} denote the simple kG -module affording the character $55_1 \in \text{Irr}(G)$. Let e_0 denote the block idempotent corresponding to the principal 3-block of kG , which is the unique block with full defect. The group G has a maximal subgroup $H \cong \mathfrak{A}_{6,23}$ such that $H \geq N_G(P)$, and inducing the non-trivial linear character $1_2 \in \text{Irr}(H)$ to G yields $e_0 \cdot \text{Ind}_H^G(1_2) = 55_1$, so that 55_1 is the character of a trivial source module. Thus V_{55} is endotrivial by Theorem 2.2.

Example 2.6 (The two faithful characters of degree 154 of $2.M_{22}$ in characteristic $\ell = 3$).

Let $G = 2.M_{22}$, let $P \in \text{Syl}_3(G)$. Let V_{154}, V_{154}^* denote the dual simple kG -modules of dimension 154 affording the characters $154_2, 154_3 \in \text{Irr}(G)$. They both belong to the faithful block B_6 . The group G has a subgroup $H \cong (2 \times \mathfrak{A}_6).2_3 \geq N_G(P)$, and inducing the linear characters $1_3, 1_4 \in \text{Irr}(H)$ to G we have $e_6 \cdot \text{Ind}_H^G(1_3) = 154_2$ and $e_6 \cdot \text{Ind}_H^G(1_4) = 154_3$, where e_6 is the block idempotent corresponding to B_6 . Therefore V_{154}, V_{154}^* are trivial source modules, thus by Theorem 2.2 it follows from the values of $154_2, 154_3$ that they are endotrivial. We note that the two faithful simple endotrivial modules V_{10}, V_{10}^* affording the characters $10_1, 10_2 \in \text{Irr}(G)$ from [16, Thm. 7.1] are also trivial source modules. Indeed $10_1 \otimes 10_1 = 45_2 + 55_1$ where 45_2 has defect zero. Whence $V_{10} \otimes_k V_{10} \cong V_{55} \oplus (\text{proj})$ and V_{10}, V_{10}^* must be trivial source since V_{55} is.

Example 2.7 (The three characters of degree 154 of HS in characteristic $\ell = 3$).

Let $G = HS$, and let V_1, V_2, V_3 denote the three simple self-dual kG -modules of dimension 154, affording the characters $154_1, 154_2, 154_3 \in \text{Irr}(G)$ respectively. Let H be a maximal subgroup of G isomorphic to M_{22} . Then $154_1|_H = 55_1 + 99_1$, where 55_1 is the character afforded by the simple endotrivial kM_{22} -module V_{55} of Example 2.5, and 99_1 has defect zero. Thus $V_1|_H \cong V_{55} \oplus (\text{proj})$ is endotrivial, and so is V_1 (see [16, Lem. 2.2]). The characters $154_2, 154_3$ lie in the non-principal block of full defect B_2 . Let e_2 be the corresponding block idempotent. Then G has a maximal subgroup $L \cong \text{U}_3(5) : 2$ with a non-trivial linear character 1_2 such that $e_2 \cdot \text{Ind}_L^G(1_2) = 154_2$. Reducing modulo 3, this

proves that V_2 is a trivial source module, hence endotrivial by Theorem 2.2. So is V_3 , the image of V_2 under the outer automorphism of G .

Example 2.8 (The four characters of degree 61776 of $6.Fi_{22}$ in characteristic $\ell = 5$).

Let $G = 6.Fi_{22}$. The characters 61776₁, 61776₃ belong to block 107 and 61776₂, 61776₄ to block 108. Let e_{107}, e_{108} denote the corresponding block idempotents respectively. Then G has two non-conjugate subgroups H and L isomorphic to $O_8^+(2).3.2$, such that $e_{107} \cdot \text{Ind}_H^G(1_H) = 61776_1$, $e_{108} \cdot \text{Ind}_H^G(1_H) = 61776_2$, $e_{107} \cdot \text{Ind}_L^G(1_L) = 61776_3$ and $e_{108} \cdot \text{Ind}_L^G(1_L) = 61776_4$. Therefore the simple reductions $V_{61776_1}, V_{61776_2}, V_{61776_3}, V_{61776_4}$ modulo 5 of these characters are trivial source modules and are endotrivial by Theorem 2.2. We recall from [16, Rem. 7.2] that Fi_{22} also has a simple trivial source endotrivial module V_{1001} affording $1001_1 \in \text{Irr}(Fi_{22})$, and $3.Fi_{22}$ has two faithful simple trivial source endotrivial modules V_{351}, V_{351}^* affording $351_1, 351_2 \in \text{Irr}(3.Fi_{22})$.

The structure of the group $TT(G)$ is not known for the sporadic groups and their covers. The above examples enable us to describe explicitly the elements of the torsion subgroup $TT(G)$ in these cases.

Corollary 2.9.

- (a) *In characteristic 3, with notation as in Examples 2.5 and 2.6 and with M an indecomposable endotrivial kM_{22} -module affording the character $154_1 \in \text{Irr}(M_{22})$ we have*

$$TT(M_{22}) = \langle [V_{55}], [M] \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

$$TT(2.M_{22}) = \langle [V_{10}], [V_{154}] \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

- (b) *In characteristic 3, with notation as in Example 2.7 we have*

$$TT(HS) = \langle [V_2], [V_3] \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

- (c) *In characteristic 5, with notation as in Example 2.8 we have*

$$TT(Fi_{22}) = \langle [V_{1001}] \rangle \cong \mathbb{Z}/2,$$

$$TT(3.Fi_{22}) = \langle [V_{1001}], [V_{351}] \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3,$$

$$TT(6.Fi_{22}) = \langle [V_{1001}], [V_{61776_1}] \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/6.$$

Proof. For $G \in \{M_{22}, 2.M_{22}, HS, Fi_{22}, 3.Fi_{22}, 6.Fi_{22}\}$, let ℓ be as in the statement, $P \in \text{Syl}_\ell(G)$ and set $N := N_G(P)$. In all cases the group $TT(G)$ injects via restriction into the group $X(N)$ of one-dimensional kN -modules, so that its elements are isomorphism classes of endotrivial trivial source modules, which are the Green correspondents of modules in $X(N)$. (See [4, §4].)

(a) If $G = 2.M_{22}$, then $P \cong 3^2$ and $X(N) \cong 2 \times 4$. The group $2.M_{22}$ has six simple trivial source endotrivial modules: k , the modules $V_{154}, V_{154}^*, V_{10}, V_{10}^*$ of Example 2.6, as well as the module V_{55} of Example 2.5 (seen as a $k[2.M_{22}]$ -module). As $TT(G)$ injects in $X(N)$, it must have 8 elements, hence $TT(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. The set of generators is obvious since both V_{10} and V_{154} are not self-dual, therefore of order 4.

The claim that $TT(G) \cong (\mathbb{Z}/2)^2$ for $G = M_{22}$ follows from the above, since $TT(G)$ is a subgroup of $TT(2.M_{22})$ via inflation and $X(N) \cong 2^2$ in this case. Using the 3-decomposition matrix of G we conclude that with $154_2, 10_1 \in \text{Irr}(2.M_{22})$ the tensor product $V_{154} \otimes_k V_{10}$ equals $M \oplus Q$, where M is an indecomposable endotrivial kG -module

affording the character $154_1 \in \text{Irr}(G)$ and Q is a projective kG -module. Whence the set of generators.

(b) For $G = HS$ we have $P \cong 3^2$, $N \cong 2 \times (3^2.SD_{16})$ and $X(N) \cong 2^3$. If e_0 denotes the principal block idempotent of kG , then $e_0 \cdot \text{Ind}_L^G(1_L) = 175_1$ with $L \cong \text{U}_3(5) : 2$ as in Example 2.7. It follows that the kG -Green correspondent of a one-dimensional module in $X(N)$ affords the character $175_1 \in \text{Irr}(G)$ but is not endotrivial by Theorem 2.2. This together with Example 2.7 forces $TT(G) = \langle [V_2], [V_3] \rangle \cong (\mathbb{Z}/2)^2$.

(c) For $G = Fi_{22}$ we have $X(N) \cong 4$. The module V_{1001} is endotrivial and trivial source, thus must be the Green correspondent of a module in $X(N)$, whereas the Green correspondents of the two other non-trivial modules in $X(N)$ are not endotrivial: this follows easily by inducing the corresponding linear characters of N to G and checking that their characters cannot satisfy the criterion of Theorem 2.2. Whence $TT(G) = \langle [V_{1001}] \rangle \cong \mathbb{Z}/2$.

For $G = 3.Fi_{22}$, $X(N) \cong 3 \times 4$ and $TT(Fi_{22}) \leq TT(G)$ via inflation. Thus in view of Example 2.8, we must have $TT(G) = \langle [V_{1001}], [V_{351}] \rangle \cong \mathbb{Z}/3 \oplus \mathbb{Z}/2$. For $G = 6.Fi_{22}$, $X(N) \cong 6 \times 4$ and $TT(3.Fi_{22}) \leq TT(G)$. Now the four faithful simple modules of dimension 61776 of Example 2.8 are not self-dual. This forces $TT(G) = \langle [V_{61776_1}], [V_{1001}] \rangle \cong \mathbb{Z}/6 \oplus \mathbb{Z}/2$. \square

3. SPECIAL LINEAR GROUPS

In this section we investigate simple endotrivial $k\text{SL}_n(q)$ -modules for $n \geq 3$, where k is a field of characteristic ℓ not dividing q . (The case of $n = 2$ or that $\ell|q$ was already considered in [16, Prop. 3.8 and Thm. 5.2].) Furthermore, we may assume that $\ell \neq 2$ by [16, Thm. 6.7]. Also, we exclude the case of cyclic Sylow ℓ -subgroups for the moment, partial results for that situation will be given in Section 3.6.

Our argument will proceed in three steps. First we treat unipotent characters, then we deal with the case when ℓ divides $q - 1$, and finally we consider the general case. But first we need to collect some auxiliary information.

3.1. Regular elements and maximal tori. Let $F : \text{GL}_n \rightarrow \text{GL}_n$ be the standard Frobenius map on the linear algebraic group GL_n over a field of characteristic p corresponding to an \mathbb{F}_q -rational structure. The conjugacy classes of F -stable maximal tori of GL_n and of SL_n are parametrized by conjugacy classes of the symmetric group \mathfrak{S}_n (and thus by partitions of n) in such a way that, if $\mathbf{T} \leq \text{GL}_n$ corresponds to $w \in \mathfrak{S}_n$ with cycle shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, then $|\mathbf{T}^F| = \prod_i (q^{\lambda_i} - 1)$, while for $\mathbf{T} \leq \text{SL}_n$ we have $(q - 1)|\mathbf{T}^F| = \prod_i (q^{\lambda_i} - 1)$. In both groups \mathbf{T} has automizer $N_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F$ isomorphic to $C_{\mathfrak{S}_n}(w)$ (see e.g. [19, Prop. 25.3]). For a prime $\ell > 2$ not dividing q we write $d_\ell(q)$ for the multiplicative order of q modulo ℓ .

Lemma 3.1. *Let $\lambda \vdash n$ be a partition, and \mathbf{T} a corresponding F -stable maximal torus of SL_n . Assume that either all parts of λ are distinct, or $q \geq 3$ and at most two parts of λ are equal. Then:*

(a) \mathbf{T}^F contains regular elements.

Now let ℓ be a prime such that some part of λ is divisible by $d := d_\ell(q)$.

- (b) *If either $d > 1$, or λ has at least two parts then \mathbf{T}^F contains ℓ -singular regular elements.*
- (c) *If either $d > 1$, or λ has at least three parts, or ℓ divides $(q-1)/\gcd(n, q-1)$ and λ has at least two parts, then \mathbf{T}^F contains ℓ -singular regular elements with non-central ℓ -part.*

Proof. Write $\lambda = (\lambda_1 \geq \dots \geq \lambda_s)$. First consider the corresponding torus $\tilde{\mathbf{T}} = \mathbf{T}Z(\mathrm{GL}_n)$ in GL_n . It is naturally contained in an F -stable Levi subgroup $\prod_i \mathrm{GL}_{\lambda_i}$ of GL_n such that $\mathbf{T}_i := \mathrm{GL}_{\lambda_i} \cap \tilde{\mathbf{T}}$ is a Coxeter torus of GL_{λ_i} . In particular, $\mathbf{T}_i^F \cong \mathbb{F}_{q^{\lambda_i}}^\times$ is a Singer cycle. Let $x_i \in \mathbf{T}_i^F$ be such that its eigenvalues are generators of $\mathbb{F}_{q^{\lambda_i}}^\times$, and $\tilde{x} = (x_1, \dots, x_s) \in \tilde{\mathbf{T}}^F$. If all λ_i are distinct, then clearly all eigenvalues of \tilde{x} are distinct, so \tilde{x} is regular. Multiplying x_1 by the inverse of $\det \tilde{x}$ yields a regular element x in $\mathbf{T}^F = \tilde{\mathbf{T}} \cap \mathrm{SL}_n(q)$. If $q \geq 3$ then there are at least two orbits of generators of $\mathbb{F}_{q^{\lambda_i}}^\times$ under the action of $\mathrm{Gal}(\mathbb{F}_{q^{\lambda_i}}/\mathbb{F}_q)$, so we may again arrange for an element with distinct eigenvalues, proving (a). Clearly, if λ_i is divisible by d then $o(x_i)$ is divisible by ℓ . If $d > 1$ or λ has at least two parts, this also holds for our modified element x , so we get (b).

Since $|Z(\mathrm{SL}_n(q))| = (n, q-1)$, (c) is clear when $d > 1$. So now assume that $d = 1$, that is, $\ell \mid (q-1)$. If λ has at least three parts, then we may arrange so that the ℓ -parts of the various x_i are not all equal, even inside $\mathrm{SL}_n(q)$, so that we obtain an element with non-central ℓ -part. The same is possible if $(q-1)/(n, q-1)$ is divisible by ℓ and λ has at least two parts. \square

We will also need some information on centralizers of semisimple elements in $\mathrm{PGL}_n = \mathrm{GL}_n/Z(\mathrm{GL}_n)$ containing F -stable maximal tori from given classes. For this, let \mathbf{H} denote an F -stable reductive subgroup of PGL_n . If \mathbf{H} contains a maximal torus of type w , then the Weyl group W_H of \mathbf{H} will have to contain a conjugate of w . Now W_H is a parabolic subgroup of the Weyl group \mathfrak{S}_n of PGL_n . Then F permutes the factors of this Young subgroup, and thus W_H^F is a product of various symmetric groups, some of them in an imprimitive action. Thus we conclude the following:

Lemma 3.2. *Let $\mathbf{H} \leq \mathrm{PGL}_n$ be a reductive subgroup containing F -stable maximal tori corresponding to cycle shapes $\lambda_1, \dots, \lambda_r$. If no intransitive or imprimitive subgroup of \mathfrak{S}_n contains elements of all these cycle shapes, then $\mathbf{H} = \mathrm{PGL}_n$.*

3.2. Unipotent characters of $\mathrm{SL}_n(q)$. We first investigate possible endotrivial simple unipotent modules of $\mathrm{SL}_n(q)$. Recall that these are naturally labelled by partitions $\lambda \vdash n$ (see e.g. [6, §13]), and we write χ_λ for the complex unipotent character with label λ . We first describe their values on regular semisimple elements.

Proposition 3.3. *Let $G = \mathrm{SL}_n(q)$. Let $t \in G$ be a regular semisimple element, and let $w \in W = \mathfrak{S}_n$ be the label of the unique F -stable maximal torus $\mathbf{T} \leq \mathrm{SL}_n$ containing t . Let $\lambda \vdash n$ be a partition, $\chi_\lambda \in \mathrm{Irr}(G)$ the corresponding unipotent character, and $\varphi_\lambda \in \mathrm{Irr}(W)$ the corresponding irreducible character of W . Then $\rho_\lambda(t) = \varphi_\lambda(w)$.*

Proof. If t is contained in a unique F -stable maximal torus \mathbf{T} of SL_n , then the character formula in [6, Prop. 7.5.3] for the Deligne–Lusztig character $R_{\mathbf{T},1}$ simplifies to

$$R_{\mathbf{T},1}(t) = \frac{1}{|T|} \sum_{\substack{g \in G \\ t \in T^g}} 1 = |N_G(\mathbf{T}) : T| = |C_W(w)|$$

(where $T = \mathbf{T}^F$), and $R_{\mathbf{T}',1}(t) = 0$ for any F -stable maximal torus \mathbf{T}' not G -conjugate to \mathbf{T} . The unipotent characters in type A coincide with the almost characters (see [6, §12.3]), so

$$\begin{aligned} \chi_\lambda(t) &= \frac{1}{|W|} \sum_{x \in W} \varphi_\lambda(x) R_{\mathbf{T}_x,1}(t) = \frac{1}{|W|} \sum_{x \in [w]} \varphi_\lambda(x) R_{\mathbf{T},1}(t) \\ &= \frac{|[w]|}{|W|} \varphi_\lambda(w) |C_W(w)| = \varphi_\lambda(w), \end{aligned}$$

where \mathbf{T}_x denotes an F -stable maximal torus parametrized by $x \in W$, and $[w]$ is the conjugacy class of w in W . \square

We will also need two closely related statements in our later treatment of exceptional groups of Lie type. For this, assume that \mathbf{G} is connected reductive with Steinberg endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$, and set $G := \mathbf{G}^F$. We consider $s \in G^*$ semisimple. The almost characters in the Lusztig series $\mathcal{E}(G, s)$ are indexed by extensions to $W_s.F$ of F -invariant irreducible characters of the Weyl group W_s of $C_{G^*}(s)$, and we denote them by $R_{\tilde{\varphi}}$, for $\tilde{\varphi}$ a fixed extension of $\varphi \in \mathrm{Irr}(W_s)^F$.

Proposition 3.4. *In the above setting, let $t \in G$ be regular semisimple and $\varphi \in \mathrm{Irr}(W_s)^F$. Then $R_{\tilde{\varphi}}(t) = 0$ unless t lies in an F -stable maximal torus \mathbf{T} such that $\mathbf{T}^* \leq C_{\mathbf{G}^*}(s)$ and \mathbf{T}^* is indexed by $wF \in W_s.F$ such that $\tilde{\varphi}(wF) \neq 0$.*

Proof. Since t is assumed to be regular, it is contained in a unique F -stable maximal torus \mathbf{T} of \mathbf{G} . The character formula in [6, Prop. 7.5.3] for the Deligne–Lusztig characters then shows that $R_{\mathbf{T},\theta}(t) = 0$ for all $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ unless $s \in \mathbf{T}^*$ up to conjugation. So now assume that $\mathbf{T}^* \leq C_{\mathbf{G}^*}(s)$ is indexed by the F -conjugacy class of $w \in W_s$. Then by definition the almost character for φ is given by

$$R_{\tilde{\varphi}}(t) = \frac{1}{|W_s|} \sum_{x \in W_s} \tilde{\varphi}(xF) R_{\mathbf{T}_x,\theta}(t) = \frac{1}{|W_s|} \sum_{x \sim w} \tilde{\varphi}(wF) R_{\mathbf{T},\theta}(t) = 0$$

unless $\tilde{\varphi}(wF) \neq 0$, as claimed, where the sum runs over the F -conjugacy class of w . \square

Proposition 3.5. *In the above setting, let $t \in G$ be regular semisimple and assume that the orders of t and s are coprime. Then $R_{\tilde{\varphi}}(t)$ is divisible by $|W(T)/W(T, \theta)|$ for all $\varphi \in \mathrm{Irr}(W_s)^F$, where $W(T)$ is the Weyl group of the unique maximal torus T containing t , and $W(T, \theta)$ is the stabilizer of $\theta \in \mathrm{Irr}(T)$ corresponding to s .*

Proof. Let \mathbf{T} be the unique F -stable maximal torus of \mathbf{G} containing t , and $T = \mathbf{T}^F$. Let (T, θ) correspond to $s \in G^*$. By [6, Prop. 7.5.3] we have

$$|R_{\mathbf{T},\theta}(t)| = \frac{1}{|T|} \sum_{g \in N_G(\mathbf{T})} \theta(t^g) = \frac{|N_G(\mathbf{T})|}{|T|} = |W(T)|,$$

since the order of the linear character θ is coprime to that of t by assumption, and so θ takes value 1 on any conjugate of t . Thus

$$R_{\tilde{\varphi}}(t) = \frac{1}{|W_s|} \sum_{x \sim w} \tilde{\varphi}(wF) R_{\mathbf{T}, \theta}(t) = \pm \frac{|W(T)| |W_s/W(T, \theta)|}{|W_s|} \tilde{\varphi}(wF) = \pm \frac{|W(T)|}{|W(T, \theta)|} \tilde{\varphi}(wF)$$

as claimed. \square

Lemma 3.6. *Let $n = 2d + r$ with $d \geq 2$ and $1 \leq r \leq d - 1$. There exists a semisimple element $t \in \mathrm{SL}_n(q)$ of order $(q^d - 1)(q^r - 1)$ with centralizer $\mathrm{GL}_2(q^d)(q^r - 1)$ in $\mathrm{GL}_n(q)$ such that the unipotent character ρ_λ with $\lambda = (d + r, r + 1, 1^{d-r-1})$ takes value $\pm q^d$ on t .*

Proof. Let $f, g \in \mathbb{F}_q[X]$ be irreducible of degrees d, r respectively with $f(0)^2 g(0) = (-1)^n$, and $t \in \mathrm{SL}_n(q)$ be a semisimple element with minimal polynomial fg and characteristic polynomial $f^2 g$. Then clearly t has centralizer $C \cong \mathrm{GL}_2(q^d)(q^r - 1)$ in $\mathrm{GL}_n(q)$, and thus is as in the statement. Since the unipotent characters of $\mathrm{SL}_n(q)$ are the restrictions of those of $\mathrm{GL}_n(q)$, we may now argue in the latter group. Now t is only contained in the two types of maximal tori T_1, T_2 of C , of orders $(q^d - 1)^2(q^r - 1)$ and $(q^{2d} - 1)(q^r - 1)$, parametrized by the partitions $\mu_1 = (d, d, r)$ and $\mu_2 = (2d, r)$ respectively. Thus the Deligne–Lusztig characters take values

$$R_{T_1, 1}(t) = \pm(q^d + 1), \quad R_{T_2, 1}(t) = \mp(q^d - 1),$$

where the signs are opposed (since $\epsilon_{T_1} = -\epsilon_{T_2}$ in the notation of [6, Prop. 7.5.3]). Furthermore, the Murnaghan–Nakayama formula gives $\chi_\lambda(\mu_1) = (-1)^{d+r+1} = -\chi_\lambda(\mu_2)$ for the values of the corresponding characters of \mathfrak{S}_n . The value $\rho_\lambda(t)$ can now be computed as in the proof of Proposition 3.3. \square

We will also need a q -analogue of Babbage’s congruence, which was first shown by Andrews [1] under stronger assumptions; here for integers $0 \leq k \leq n$ we set

$$\begin{bmatrix} n \\ k \end{bmatrix}_x := \prod_{i=1}^k \frac{x^{n-k+i} - 1}{x^i - 1} \in \mathbb{Z}[x].$$

Lemma 3.7. *Let x be an indeterminate and $d, h \geq 2$. Then in $\mathbb{Z}[x]$ we have*

$$\begin{bmatrix} hd - 1 \\ d - 1 \end{bmatrix}_x \equiv x^{(h-1)\binom{d}{2}} \pmod{\Phi_d(x)^2}.$$

Proof. When d is a prime, then $\Phi_d(x) = (x^d - 1)/(x - 1)$, and in that case the claim is proved in [1, Thm. 2]. But inspection shows that the argument given there is valid for all $d \geq 2$ if congruences are considered just modulo $\Phi_d(x)^2$ instead of $(x^d - 1)^2/(x - 1)^2$. \square

Proposition 3.8. *Let χ be a non-trivial unipotent character of $G = \mathrm{SL}_n(q)$, $n \geq 3$, and $2 \neq \ell \nmid q$ a prime such that $n \geq 2d$ where $d := d_\ell(q)$. Then χ is not the character of a simple endotrivial module for any central factor group of G .*

Proof. Unipotent characters have the centre of G in their kernel, hence we may see χ as a character of any central factor group S of G . Let $\lambda \vdash n$ denote the label for χ . First note that we have $\lambda \neq (n)$, since (n) parametrizes the trivial character of G . The Steinberg character, parameterized by (1^n) , can never be the character of a simple endotrivial kS -module, for it has degree $q^{\binom{n}{2}}$, so can be congruent to ± 1 modulo $|S|_\ell$ only when $n = \ell = 3$,

$q \equiv 4, 7 \pmod{9}$. But by [14, Tab. 4] the Steinberg character is reducible modulo 3 in this case.

If χ is endotrivial, then in particular its values on ℓ -singular elements have to be of absolute value one (see [16, Cor. 2.3]). We start with the case that $d = 1$, so $\ell | (q - 1)$ (and hence $q \geq 4$ as $\ell \neq 2$). First assume that moreover $(q - 1)/(n, q - 1)$ is divisible by ℓ , so $|Z(G)| = (n, q - 1)$ is not divisible by the full ℓ -part of $q - 1$. Let T be a maximal torus of G corresponding to an element $w \in \mathfrak{S}_n$ of cycle shape $(n - 1)(1)$. By Lemma 3.1, T contains a regular element x of G with non-central ℓ -part. Then $\chi(x) = \varphi_\lambda(w)$ by Proposition 3.3. By the Murnaghan–Nakayama formula the latter is zero unless λ has an $n - 1$ -hook. Similarly, with a maximal torus corresponding to the cycle shape $(n - 2)(2)$ we see that λ has to possess an $n - 2$ -hook. When $n \geq 6$ we may also argue that λ has an $n - 3$ -hook, using the cycle shape $(n - 3)(3)$. The only partitions possessing all three types of hooks are (n) and (1^n) , which we excluded above. When $n \leq 5$, there are also the possibilities $\lambda = (2, 2), (3, 2)$ and $(2^2, 1)$. From the known values of unipotent characters in $\mathrm{SL}_4(q)$ and $\mathrm{SL}_5(q)$ provided in Chevie [9] it follows that all three characters vanish on the product of a Jordan block of size $n - 1$ with a commuting non-central ℓ -element.

Next assume that $d = 1$, but now $(q - 1)/(n, q - 1)$ is prime to ℓ , so that in particular ℓ divides n . Thus if $n \leq 6$, we only need to consider the pairs $(n, \ell) \in \{(3, 3), (5, 5), (6, 3)\}$. If ℓ divides $|Z(S)|$, then χ cannot take absolute value one values on central ℓ -elements. Otherwise, it can be checked from the known character tables of unipotent characters [9] that there are no examples of degree congruent to ± 1 modulo $|S|_\ell$. For $n \geq 7$, arguing with maximal tori corresponding to cycle shapes $(n - 2)(1)^2, (n - 3)(2)(1)$, we see by Lemma 3.1 that λ must possess $n - 2$ - and $n - 3$ -hooks, whence λ or its conjugate partition is one of $(n), (n - 1, 1), (n - 3, 3), (n - 4, 2^2)$ (recall that $q \geq 4$). Now removing an $n - 3$ -hook from the second and third of these partitions or their conjugate partitions leaves a 2-core, so the unipotent characters labelled by these vanish on regular semisimple elements of a torus of type $(n - 3)(2)(1)$. Finally, the character labelled by the partition $(n - 4, 2^2)$ vanishes on regular elements in a torus of type $(n - 4)(3)(1)$, so is not endotrivial. This completes the discussion for $d = 1$.

So now let us assume that $d \geq 2$. Write $n = ad + r$ with $0 \leq r < d$. Let T be a maximal torus of G corresponding to an element $w \in \mathfrak{S}_n$ of cycle shape $(n - r)(r)$. As before by Lemma 3.1 and Proposition 3.3 we conclude that there is an ℓ -singular regular element $x \in T$ such that $\chi(x) = \varphi_\lambda(w)$. By the Murnaghan–Nakayama formula the latter is zero unless λ has an $n - r$ -hook such that the partition obtained by removing that hook is an r -hook. Now first assume that $r \geq 3$. Then similarly, with a torus corresponding to an element $w \in \mathfrak{S}_n$ of cycle shape $(n - r)(r - 1)(1)$ we see that the remaining partition has to possess an $r - 1$ -hook. The only partitions with that property are (up to taking the conjugate partition), $\lambda = (n - r - s + 1, 2^s, 1^{r-s-1})$ for some $0 \leq s \leq r - 1$, and $\lambda = (n - r - s, r + 1, 1^{s-1})$ for some $1 \leq s \leq n - 2r - 1$. Next, consider tori corresponding to the cycle shape $(n - d)(d)$, which again contain ℓ -singular regular elements. This forces λ to possess an $n - d$ -hook such that removing it leaves a d -hook. For the first type of partition, this is only possible when $s = 0$, so $\lambda = (n - r + 1, 1^{r-1})$ is a hook. For the second type, it forces $s = d - r$, whence $\lambda = (n - d, r + 1, 1^{d-r-1})$. So, up to taking conjugates, λ is one of $(n - r + 1, 1^{r-1})$ or $(n - d, r + 1, 1^{d-r-1})$.

Next we consider the situation where $r \leq 2$. First assume that $r = 0$. Then λ has an n -hook, so it is a hook. The cycle shapes $(n-d)(d)$ and $(n-d)(d-1)(1)$ then show that $\lambda = (n-d+1, 1^{d-1})$ or the conjugate partition.

If $r = 1$ then λ has an $n-r = n-1$ -hook, so $\lambda = (n-s, 2, 1^{s-2})$ for some $2 \leq s \leq n/2$. The cycle shape $(n-d)(d)$ then forces $\lambda = (n-d, 2, 1^{d-2})$.

If $r = 2$ (so $d \geq 3$) then λ has an $n-2$ -hook, so $\lambda = (n-s, 3, 1^{s-3})$ for some $3 \leq s \leq n-3$ or $\lambda = (n-s, 2^2, 1^{s-4})$ for some $4 \leq s \leq n-2$, or $\lambda = (n-1, 1)$ or $\lambda = (n-2, 2)$. Again, the cycle shape $(n-d)(d)$ rules out the last case; the first and second case are only possible for $s = d$, so we are left with $(n-d, 3, 1^{d-3})$ (with $d \geq 3$), $(n-d, 2^2, 1^{d-4})$ (with $d \geq 4$), and $(n-1, 1)$. This concludes our discussion of the various possibilities for r .

Now allow r to be arbitrary again. If $n \geq 3d$ and $r > 0$, consider the cycle shape $(n-d-r)(d)(r)$. Since $(n-d, r+1, 1^{d-r-1})$ does not have an $n-d-r$ -hook, this case is out. The cycle shape $(n-d-1)(d)(1)$ shows that $(n-r+1, 1^{r-1})$ cannot occur, and neither can $(n-d+1, 1^{d-1})$ for $n \neq 2d$ nor $(n-d, 2^2, 1^{d-4})$ for $r = 2$ and $n > 2d+2$ occur. Finally, the cycle shape $(d+1)(d)(1)$ rules out $(d+2, 2^2, 1^{d-4})$. So at this point, up to taking conjugates λ can only be one of $(d+r, r+1, 1^{d-r-1})$ with $d > r > 0$, or $(d+1, 1^{d-1})$. In particular $n < 3d$ and so G has ℓ -rank at most 2.

We deal with the cases individually. For $\lambda = (d+1, 1^{d-1})$ the degree of χ_λ is

$$\chi_\lambda(1) = q^{\binom{d}{2}} \begin{bmatrix} 2d-1 \\ d-1 \end{bmatrix}_q$$

(see e.g. [6, 13.8]), so that by Lemma 3.7 we have that

$$\chi_\lambda(1) \equiv q^{d(d-1)} \pmod{\Phi_d(q)^2}.$$

Since $\Phi_d(q)^2$ divides $|G|$, we see that $\chi_\lambda(1)$ is not congruent to ± 1 modulo $|G|_\ell$. The degree of the unipotent character labelled by the conjugate partition only differs by a power of q , so the same argument applies.

It follows from [11, Thm. 4.15] that the Specht module of the Hecke algebra H of type A_{n-1} indexed by the d -regular partition $\lambda = (d+r, r+1, 1^{d-r-1})$ or its conjugate is reducible if q is specialized to a d th root of unity, for $r = 1, \dots, d-2$. Since the decomposition matrix of H embeds into that of $\mathrm{SL}_n(q)$ (see e.g. [10, Thm. 4.1.14]), the corresponding unipotent characters are reducible modulo ℓ , so do not correspond to simple endotrivial modules. (Alternatively, one could also appeal to Lemma 3.6 for the partition $(d+r, r+1, 1^{d-r-1})$, but not for its conjugate.) It remains to consider $\lambda = (d+r, r+1, 1^{d-r-1})$ and its conjugate for $r = d-1$, that is, $\lambda = (2d-1, d)$. Then

$$\chi_\lambda(1) = q^d \begin{bmatrix} 3d-1 \\ d-1 \end{bmatrix}_q \equiv q^{d^2} \pmod{\Phi_d(q)^2}$$

again by Lemma 3.7 (resp. times some power of q for the conjugate partition), and we are done as before. \square

3.3. $\mathrm{SL}_n(q)$ with $\ell|(q-1)$. We next investigate arbitrary irreducible characters of $\mathrm{SL}_n(q)$ when $\ell|(q-1)$.

Proposition 3.9. *Let $G = \mathrm{SL}_n(q)$ with $n \geq 3$ and $2 \neq \ell \nmid q$ a prime with $d_\ell(q) = 1$. Then no central factor group of G has a non-trivial simple endotrivial module in characteristic ℓ .*

Proof. Let S be a central factor group of G , and V a simple endotrivial kS -module, where k is of characteristic ℓ . Then V lifts to a $\mathbb{C}S$ -module by [16, Thm. 1.3]. Let $\chi \in \text{Irr}(S)$ denote the character of such a lift. We may and will consider χ as an irreducible character of G . Thus, χ lies in the Lusztig series $\mathcal{E}(G, s)$ of a semisimple element s in the dual group $G^* = \text{PGL}_n(q)$. By [16, Prop. 6.3] then s must lie in a maximal torus containing a Sylow 1-torus (as $d_\ell(q) = 1$). But Sylow 1-tori are maximal tori of G^* , so s is contained in a maximally split torus. Thus, $C_{G^*}^\circ(s)$ is a (1-split) Levi subgroup of G^* . For $s = 1$ we obtain unipotent characters, for which there is no example by Proposition 3.8, so from now on let $s \neq 1$.

As in the proof for unipotent characters, first assume that $(n, q-1)$ does not contain the full ℓ -part of $q-1$. Note that our assumption $\ell \neq 2$ forces $q \geq 4$. When $n \geq 4$ let T_1, T_2 denote maximal tori of G corresponding to the cycle shapes $(n-1)(1)$ and $(n-2)(2)$. By Lemma 3.1 both tori do contain regular elements with non-central ℓ -part. Thus by [16, Prop. 6.4], if V is endotrivial then $C_{G^*}^\circ(s)$ contains conjugates of the dual tori T_1^*, T_2^* . But by Lemma 3.2 there does not exist a proper 1-split Levi subgroup of $\text{PGL}_n(q)$ containing these two types of tori, whence we are done.

Now let's consider the case $n = 3$ which was excluded above. Here the known character table of G (see [9]) shows that at most the characters of degree $\chi(1) = q(q^2 + q + 1)$ might satisfy the necessary conditions about values on ℓ -singular elements. As $\chi(1) \equiv 3 \pmod{(q-1)}$, we must have $\ell = 2$ which was excluded.

It remains to consider the case that $(n, q-1)$ is divisible by the full ℓ -part of $q-1$. (So, in particular, $\ell|n$ and hence $n \neq 4$). Here, for $n \geq 7$ we argue using maximal tori corresponding to cycle shapes $(n-2)(1)^2, (n-4)(3)(1)$ and $(n-4)(2)^2$, which all contain regular elements with non-central ℓ -part, to rule out all possible proper 1-split Levi subgroups as $C_{G^*}^\circ(s)$.

Finally we deal with the cases $n = 3, 5, 6$. For $n = 6$ (and so $\ell = 3$) it can be checked by Lusztig's parametrization or from the explicitly known lists that all non-unipotent characters either have degree divisible by 3, or are of defect zero for a Zsigmondy prime divisor r of $q^2 + 1$. Since a torus of type $(4)(1)^2$ contains elements with non-central ℓ -part and of order divisible by r , the latter characters vanish on such elements, hence cannot belong to endotrivial modules. For $n = 5$, the only non-unipotent characters of degree not divisible by $\ell = 5$ are the five characters of degree $\Phi_2^2 \Phi_3 \Phi_4 \Phi_5 / 5$. But these are of r -defect zero for a Zsigmondy prime divisor of $q^3 - 1$, and since a maximal torus of type $(3)(1)^2$ contains regular elements with non-central ℓ -part which are r -singular, these characters do not lead to examples. Finally, when $n = 3$ the only non-unipotent characters of degree not divisible by $\ell = 3$ are the three characters of degree $\Phi_2 \Phi_3 / 3$, where $q \equiv 1 \pmod{3}$, but these are reducible modulo 3 by [14, Tab. 4]. \square

3.4. The general case.

Theorem 3.10. *Let S be a central factor group of $\text{SL}_n(q)$ with $n \geq 3$. Let $\ell \nmid q$ be such that the Sylow ℓ -subgroups of S are non-cyclic. Then S has no non-trivial simple endotrivial module in characteristic ℓ .*

Proof. As in the proof of Proposition 3.9 we may argue with complex irreducible characters of $G = \text{SL}_n(q)$. Let ℓ be a prime divisor of $|G|$ and set $d := d_\ell(q)$. We may assume that ℓ is odd by [16, Thm. 6.7]. If $d = 1$ then the claim is contained in Proposition 3.9. So now

assume that $d > 1$, and write $n = ad + r$ with $0 \leq r < d$. Since the Sylow ℓ -subgroups of G are non-cyclic we have $a \geq 2$, so $n \geq 2d$. Let $\chi \in \text{Irr}(G)$ be the character of an endotrivial simple kG -module. Then χ lies in some Lusztig series $\mathcal{E}(G, s)$, where $s \in G^* = \text{PGL}_n(q)$ is semisimple. Now G contains maximal tori T_1, T_2, T_3 parametrized by the partitions $(n-d, d)$, $(n-r, r)$ and $(n-d-1, d, 1)$, all of which contain ℓ -singular regular semisimple elements by Lemma 3.1. As χ is endotrivial, it cannot vanish on these elements. But then by [16, Prop. 6.4] the centralizer $C_{G^*}(s)$ contains maximal tori of these three types. By Lemma 3.2 the only reductive subgroup of G^* containing tori of all three types is G^* itself, so $s = 1$ and χ is a unipotent character. Since we have $n \geq 2d$, the claim now follows from Proposition 3.8. \square

3.5. Exceptional covering groups. We now discuss the characters of exceptional covering groups of classical groups.

Proposition 3.11. *Let G be an exceptional covering group of a simple group of classical Lie type and ℓ a prime such that the Sylow ℓ -subgroups of G are non-cyclic. If V is a faithful simple endotrivial kG -module, then $\ell = 3$ and (G, V) are as listed in Table 1.*

TABLE 1. Faithful simple endotrivial modules for exceptional covering groups of simple groups of classical Lie type

G	3-rank	$\dim V$
$2.\text{L}_3(4)$	2	10, 10
$4_1.\text{L}_3(4)$	2	8, 8, 8, 8
$4_2.\text{L}_3(4)$	2	28, 28, 28, 28

Proof. Let $\chi \in \text{Irr}(G)$ be the character afforded by the lift of a faithful simple endotrivial kG -module V . Using [16, Lem. 2.2, Cor. 2.3], the known ordinary character tables and decomposition matrices (see [22]) we obtain the list of candidates for χ as listed in Table 1. In particular, we always have $\ell = 3$, and none of the exceptional covering groups of $\text{U}_4(2)$, $\text{U}_4(3)$, $\text{U}_6(2)$, $\text{Sp}_6(2)$, $\text{O}_7(3)$ and $\text{O}_8^+(2)$ leads to a candidate character.

If $(G, \chi(1)) = (4_1.\text{L}_3(4), 8)$, then $\chi \otimes \chi^*$ has one trivial constituent and one constituent of defect zero. Hence reducing modulo 3 yields $V \otimes_k V^* \cong k \oplus (\text{proj})$. If $G = 2.\text{L}_3(4)$ and $\chi(1) = 10$, then $G \leq 2.M_{22}$ and $\chi = \psi|_G$ where $\psi \in \text{Irr}(2.M_{22})$ is a faithful character of degree 10, which is afforded by a simple endotrivial $k2.M_{22}$ -module M by [16, Thm. 7.1]. Thus $V = M|_G$ is endotrivial by [16, Lem. 2.2]. If $(G, \chi(1)) = (4_2.\text{L}_3(4), 28)$, then V is endotrivial by Theorem 2.2 as it is the Green correspondent of a one-dimensional $kN_G(P)$ -module for $P \in \text{Syl}_3(G)$. Indeed, there are non-trivial linear characters 1_a and 1_b of $N_G(P)$ such that

$$\text{Ind}_{N_G(P)}^G(1_a) = \text{Ind}_{N_G(P)}^G(1_b) = 28_3 + 28_5 + 64_3 + 80_1 + 80_3.$$

Because 1_a and 1_b are not dual to each other, it follows that their kG -Green correspondents (which are trivial source modules) afford distinct complex characters. Therefore $\text{Ind}_{N_G(P)}^G(1_a)$ and $\text{Ind}_{N_G(P)}^G(1_b)$ are both the characters of decomposable modules. Hence it follows from Green correspondence and the possible degrees and values of trivial source

modules given by Lemma 2.1 that $28_3, 28_5$ are the characters afforded by the kG -Green correspondents of 1_a and 1_b . Similarly for 28_4 and 28_6 as they are dual to 28_3 and 28_5 respectively. \square

3.6. Cyclic blocks. To conclude the investigation of $\mathrm{SL}_n(q)$ we discuss cyclic blocks. Endotrivial modules have dimension prime to ℓ , hence lie in ℓ -blocks of full defect. Therefore if a simple endotrivial module of a finite group G lies in a cyclic block, then the Sylow ℓ -subgroups of G are cyclic. We let $\mathrm{se}(G)$ denote the number of isomorphism classes of simple endotrivial modules and $\mathrm{sb}(G)$ the number of blocks containing simple endotrivial modules.

Proposition 3.12. *Let $G = \mathrm{SL}_n(q)$ be quasi-simple with $n \geq 2$. Let $\ell \nmid q$ be a prime such that the Sylow ℓ -subgroups of G are cyclic, and let $d := d_\ell(q)$. Then $\ell > 2$ and the number $\mathrm{sb}(G)$ is as follows.*

- (a) *If $d = 1$, then $n = 2$, and $\mathrm{sb}(G) = 2$ if q is odd and $\mathrm{sb}(G) = 1$ if q is even.*
- (b) *If $d > 1$ then*

$$\mathrm{sb}(G) = \begin{cases} \gcd(q-1, n) & \text{if } n = d, \\ 2 & \text{if } n = d+2 \text{ and } q = 2, \\ q-1 & \text{else.} \end{cases}$$

Proof. First G does not have cyclic Sylow 2-subgroups (by the Brauer–Suzuki theorem), hence $\ell > 2$. Second if $\ell \mid (q-1)$, then $n = 2$ and the simple endotrivial kG -modules are classified in [16, Prop. 3.8]. Therefore we may assume $d > 1$ and we have $n < 2d$.

Let $P \in \mathrm{Syl}_\ell(G)$. By [16, Lem. 3.2], the number $\mathrm{sb}(G)$ is $\frac{1}{e}|X(H)| = \frac{1}{e}|H/[H, H]|_{\ell'}$, where e denotes the inertial index of the principal block of G and $H = N_G(\langle u \rangle) = N_G(P)$ for $u \in P$ an element of order ℓ . As $d > 1$, any ℓ -block of G is covered by a cyclic defect block of $\mathrm{GL}_n(q)$, hence e is equal to the inertial index of the principal block of $\mathrm{GL}_n(q)$, and it follows from [8, (9)] that $e = d$. Finally $|X(H)|$ is calculated in [5, Thm. 1.2] as follows: if $n = d$, then $|X(H)| = c'd$ with $c' = \gcd(q-1, (q^d-1)/(q-1))$ which equals $\gcd(q-1, d)$, whereas $|X(H)| = d(q-1)$ if $n = d+f$ with $f > 0$, with the exception that $|X(H)| = 2d$ if $n = d+2$ and $q = 2$. The claim follows. \square

We recall from [16, §3] that a block B of kG containing endotrivial modules has inertial index equal to that of the principal block. Moreover simple endotrivial kB -modules coincide with the non-exceptional end nodes of its Brauer tree $\sigma(B)$. A description of the Brauer trees for $\mathrm{SL}_n(q)$ with $d_\ell(q) > 1$ is provided by [20].

Corollary 3.13. *Let $G = \mathrm{SL}_n(q)$, $n > 2$, with cyclic $P \in \mathrm{Syl}_\ell(G)$, and let $d = d_\ell(q)$ (and hence $d > 1$). Let B be a block of kG containing a simple endotrivial module V .*

- (a) *If $1 < d < n$, then V is the restriction of a simple endotrivial module of $\mathrm{GL}_n(q)$, and $\sigma(B)$ is an open polygon with exceptional vertex sitting at one end. Moreover,*

$$\mathrm{se}(G) = \begin{cases} 2(q-1) & \text{if } d = \ell - 1 \text{ and } |P| = \ell, \\ q-1 & \text{else,} \end{cases}$$

unless $q = 2$ and $n = d + 2$, in which case

$$\text{se}(G) = \begin{cases} 4 & \text{if } d = \ell - 1 \text{ and } |P| = \ell, \\ 2 & \text{else.} \end{cases}$$

- (b) If $d = n$ and $Z(G) = \{1\}$, then B is the principal block B_0 and V is the restriction of a simple endotrivial module of $\text{GL}_n(q)$. Moreover,

$$\text{se}(G) = \begin{cases} 2 & \text{if } d = \ell - 1 \text{ and } |P| = \ell, \\ 1 & \text{else.} \end{cases}$$

In particular, if $d \neq \ell - 1$, or $|P| \neq \ell$, then V is trivial.

- (c) If $d = n$ and $Z(G) \neq \{1\}$, then either $B = B_0$ and V is the restriction of a simple endotrivial module of $\text{GL}_n(q)$, or $B \neq B_0$ and $\sigma(B)$ is a star with exceptional vertex in the middle and $d/e_{\tilde{B}} > 1$ equal-length branches, where $e_{\tilde{B}}$ is the inertial index of any block \tilde{B} of full defect of $\text{GL}_n(q)$ covering B .

Proof. Set $\tilde{G} := \text{GL}_n(q)$. First we claim that the kernel K of the restriction map $\text{Res}_{\tilde{G}}^G : T(\tilde{G}) \rightarrow T(G) : [M] \mapsto [M|_G]$ is $X(\tilde{G})$, the group of one-dimensional $k\tilde{G}$ -modules. As G is a perfect group, certainly $X(\tilde{G}) \subseteq K$. Now assume that M is an indecomposable endotrivial $k\tilde{G}$ -module such that $M|_G = k \oplus (\text{proj})$. Then, as $d > 1$, M is G -projective so that $M|_G \cong k\tilde{G}$ and in turn by the Mackey formula, $M|_G$ is a summand of

$$k\tilde{G}|_G \cong \bigoplus_{x \in [\tilde{G}/G]} {}^x k \cong \bigoplus_{x \in [\tilde{G}/G]} k.$$

Comparing both decompositions of $M|_G$ yields $M|_G \cong k$ and so $\dim_k M = 1$.

Assume that $1 < d < n$. In this situation $|T(\tilde{G})| = (q-1)|T(G)|$ by [5, Thm. 1.2(d)(ii)]. Therefore, as $K = X(\tilde{G}) \cong C_{q-1}$, the map $\text{Res}_{\tilde{G}}^G$ is surjective. Thus any block B of kG containing a simple endotrivial module V is covered by a block \tilde{B} of $k\tilde{G}$ containing an endotrivial module. By [16, Thm. 3.7], simple endotrivial modules lying in B and \tilde{B} coincide with the non-exceptional end nodes of $\sigma(B)$ and $\sigma(\tilde{B})$ respectively. By [8, Thm. C], $\sigma(\tilde{B})$ is an open polygon with exceptional vertex sitting at one end. So let \tilde{V} be the simple endotrivial module at the other end of $\sigma(\tilde{B})$, and let $\tilde{\chi} \in \text{Irr}(\tilde{B})$ be the corresponding character. By [20, Thm. 1], $\sigma(B)$ is a star obtained by unfolding $\sigma(\tilde{B})$ around its exceptional vertex, so that $\tilde{\chi}|_G$ is a sum of $d/e_{\tilde{B}}$ irreducible constituents labelling the end vertices of $\sigma(B)$. Now $\tilde{V}|_G = V_0 \oplus (\text{proj})$, where on the one hand V_0 is indecomposable endotrivial by [16, Lem. 2.2], and on the other hand V_0 is simple since $G \triangleleft \tilde{G}$. This forces $\tilde{V}|_G = V_0$ and $\sigma(B)$ to be an open polygon with exceptional vertex at the end (i.e., a star with one branch). The statements on $\text{se}(G)$ are then straightforward from Proposition 3.12 and [16, Thm. 3.7].

Now assume $d = n$. The principal block B_0 is covered by the principal block \tilde{B}_0 of $k\tilde{G}$. Therefore the above argument applies again and shows that $\sigma(B_0)$ is an open polygon with exceptional vertex at one end and any simple endotrivial kG -module is the restriction of a simple endotrivial $k\tilde{G}$ -module. If $Z(G) = \{1\}$, then by Proposition 3.12 the unique block containing simple endotrivial modules is B_0 . Whence $\text{se}(G) = 2$ if the exceptional multiplicity is one, i.e., $d = \ell - 1$ and $|P| = \ell$, and $\text{se}(G) = 1$ otherwise. If $Z(G) \neq \{1\}$,

then $|T(\tilde{G})| = 2d(q-1)$ by [5, Thm. 1.2(d)(i)], so that $\text{sb}(\tilde{G}) = \frac{1}{2d}|T(\tilde{G})| = q-1$ by [16, Lem. 3.2]. As $K = X(\tilde{G}) \cong C_{q-1}$, it follows that each block \tilde{B} of $k\tilde{G}$ containing an endotrivial module contains a one-dimensional $k\tilde{G}$ -module and covers B_0 . Therefore, if B is a non-principal block of kG , then it is covered by blocks of $k\tilde{G}$ containing no endotrivial modules. Then [20, Thm. 1] forces $\sigma(B)$ to be a star with exceptional vertex in the middle and $d/e_{\tilde{B}} > 1$ branches, for if $d/e_{\tilde{B}}$ were 1, then again by a similar argument to that given in the case $1 < d < n$, a simple module at the end of $\sigma(\tilde{B})$ would be endotrivial. \square

4. SPECIAL UNITARY GROUPS

In this section we classify simple endotrivial modules of special unitary groups. Many arguments can be copied from the case of special linear groups. As before, we first study unipotent characters.

4.1. Unipotent characters of $\text{SU}_n(q)$ with $n \geq 3$. Let $F : \text{GL}_n \rightarrow \text{GL}_n$ be the twisted Steinberg endomorphism with $\text{GL}_n^F = \text{GU}_n(q)$. Again the conjugacy classes of F -stable maximal tori of GL_n and of SL_n are parametrized by partitions of n in such a way that, if $\mathbf{T} \leq \text{GL}_n$ corresponds to the partition λ , then $|\mathbf{T}^F| = \prod_i (q^{\lambda_i} - (-1)^{\lambda_i})$, while for $\mathbf{T} \leq \text{SL}_n$ we have $|\mathbf{T}^F| = \prod_i (q^{\lambda_i} - (-1)^{\lambda_i}) / (q+1)$. In both cases \mathbf{T} has automizer isomorphic to $C_{\mathfrak{S}_n}(w)$. We need some information on regular elements.

Lemma 4.1. *Let $\lambda \vdash n$ be a partition, and \mathbf{T}^F a corresponding maximal torus of $\text{SU}_n(q)$. Assume that either all parts of λ are distinct, or $q \geq 3$ and at most two parts of λ are equal. Then:*

(a) \mathbf{T}^F contains regular elements.

Now let ℓ be a prime such that some part of λ is divisible by $d := d_\ell(-q)$.

(b) If either $d > 1$, or λ has at least two parts then \mathbf{T}^F contains ℓ -singular regular elements.

(c) If either $d > 1$, or λ has at least three parts, or ℓ divides $(q+1)/\gcd(n, q+1)$ and λ has at least two parts, then \mathbf{T}^F contains ℓ -singular regular elements with non-central ℓ -part.

Proof. Write $\lambda = (\lambda_1 \geq \dots \geq \lambda_s)$. As in the proof of Lemma 3.1 the corresponding torus $\tilde{\mathbf{T}}$ in GL_n is a direct product of F -stable factors \mathbf{T}_i contained in an F -stable Levi subgroup $\prod_i \text{GL}_{\lambda_i}$, such that \mathbf{T}_i^F is isomorphic to the cyclic subgroup N_i of order $q^{\lambda_i} - (-1)^{\lambda_i}$ of $\mathbb{F}_{q^{2\lambda_i}}^\times$. Choose $x_i \in \mathbf{T}_i^F$ such that its eigenvalues are generators of N_i , and set $\tilde{x} = (x_1, \dots, x_s) \in \tilde{\mathbf{T}}^F$. If all λ_i are distinct, then all eigenvalues of \tilde{x} are distinct, so \tilde{x} is regular. Multiplying x_1 by the inverse of $\det \tilde{x}$ yields a regular element x in $\mathbf{T}^F = \tilde{\mathbf{T}} \cap \text{SU}_n(q)$. If $q \geq 3$ (or $\lambda_i \neq 2$) then there are at least two orbits of generators of N_i under the action of the Galois group, so we may still arrange for an element with distinct eigenvalues, which proves (a).

Clearly, if λ_i is divisible by d then $o(x_i)$ is divisible by ℓ . If $d > 1$ or λ has at least two parts, this also holds for our modified element x , so we get (b).

Since $|Z(\text{SU}_n(q))| = (n, q+1)$, (c) is clear when $d > 1$. So now assume that $\ell | (q+1)$. If λ has at least three parts, then we may arrange so that the ℓ -parts of the various x_i are not all equal, even inside $\text{SU}_n(q)$, so that we obtain an element with non-central ℓ -part, and similarly if $(q+1)/(n, q+1)$ is divisible by ℓ and λ has at least two parts. \square

The assertion and proof of Proposition 3.3 on values of unipotent characters on regular semisimple elements remain valid up to sign by replacing class functions on the Weyl group by F -class functions.

Proposition 4.2. *Let χ be a non-trivial unipotent character of $G = \mathrm{SU}_n(q)$, $n \geq 3$, and $2 \neq \ell \nmid q$ a prime such that $n \geq 2d$ where $d := d_\ell(-q)$. Then χ is not the character of a simple endotrivial module for any central factor group S of G .*

Proof. From the known table of unipotent characters [9] we conclude that there are no non-trivial endotrivial unipotent characters when $n \leq 6$. So assume that $n \geq 7$. Let $\lambda \vdash n$ denote the label for χ . We may assume that $\lambda \neq (n)$, since (n) parametrizes the trivial character of G . The Steinberg character, parameterized by (1^n) , has degree $q^{\binom{n}{2}}$, so can be congruent to ± 1 modulo $|G|_\ell$ only when $n = \ell = 3$, $q \equiv 2, 5 \pmod{9}$, but again the Steinberg character is reducible modulo 3 in this situation by [13, Lemma 4.3].

First assume that $d = 1$. We then argue precisely as in the proof of Proposition 3.8 to deduce that χ vanishes on some ℓ -singular element with non-central ℓ -part, so χ is not endotrivial.

Similarly, when $d \geq 2$ we write $n = ad + r$ with $0 \leq r < d$. We again follow the arguments in the proof of Proposition 3.8, using Lemma 4.1 and the analogue of Proposition 3.3 to conclude that λ must be one of $(d+r, r+1, 1^{d-r-1})$ with $0 < r < d$, or $(d+1, 1^{d-1})$, up to taking conjugates. The degree of χ_λ , for λ one of $(d+1, 1^{d-1})$ or $(2d-1, d)$, is obtained from the one of the corresponding character in $\mathrm{SL}_n(q)$ by replacing q by $-q$ (see [6, 13.8]), so the same congruences as in the proof of Proposition 3.8 show that these two characters cannot be endotrivial. Next, by the Ennola dual of Lemma 3.6 the unipotent character indexed by $(d+r, r+1, 1^{d-r-1})$, with $0 < r < d-1$, takes value $\pm q^d$ on a semisimple element of order a multiple of $q^d - (-1)^d$ with centralizer $A_2((-q)^d)(q^r - (-1)^r)$, so cannot be endotrivial. Finally, by Lusztig [18, Thm. 11.2] the unipotent character indexed by the conjugate partition $(d-r+1, 2^r, 1^{d-1})$ vanishes on any element with unipotent part contained in the closure of its unipotent support of Jordan type $(d+r, r+1, 1^{d-r-1})$. Now $\mathrm{SU}_n(q)$ contains elements of order $(q^d - (-1)^d)/(q+1)$ which centralize a unipotent Jordan block of size $d+r$, and that is contained in the closure of the class with Jordan type $(d+r, r+1, 1^{d-r-1})$. This deals with the last open case. \square

4.2. $\mathrm{SU}_n(q)$ with $\ell \mid (q+1)$. We next consider the case that $d_\ell(-q) = 1$. Here, we actually find examples of simple endotrivial modules.

Proposition 4.3. *Let $G = \mathrm{SU}_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$, and ℓ a prime divisor of $|G|$ with $d_\ell(-q) = 1$. Let V be a non-trivial simple kS -module for some central factor group S of G with k of characteristic ℓ . Then V is endotrivial if and only if $S = \mathrm{U}_3(q)$, $\ell = 3$, $q \equiv 2, 5 \pmod{9}$ and $\dim V = (q-1)(q^2 - q + 1)/3$.*

Proof. Let V be a simple endotrivial kS -module. We may and will consider V as a kG -module. Let $\chi \in \mathrm{Irr}(G)$ be the complex character of a lift of V . Then χ lies in the Lusztig series $\mathcal{E}(G, s)$ of a semisimple element $s \in G^* = \mathrm{PGU}_n(q)$. By [16, Prop. 6.3] then s must lie in a maximal torus containing a Sylow 2-torus (as $d_\ell(-q) = 1$). But Sylow 2-tori are maximal tori of G^* , so $C_{G^*}^\circ(s)$ is a 2-split Levi subgroup of G^* . The case of unipotent characters where $s = 1$ does not provide examples by Proposition 4.2, so assume that $s \neq 1$.

We now mimic the proof of Proposition 3.9. If $(n, q+1)$ does not contain the full ℓ -part of $q+1$, the case that $n > 3$ can be ruled out by consideration of maximal tori of G corresponding to the cycle shapes $(n-1)(1)$ and $(n-2)(2)$ (which by Lemma 4.1 both contain regular elements with non-central ℓ -part). When $n = 3$ the known character table of G (see [9]) shows that at most the characters χ of degree $q(q^2 - q + 1)$ might satisfy the necessary conditions about values on ℓ -singular elements. As $\chi(1) \equiv -3 \pmod{(q+1)}$, we must have $\ell = 2$ which was excluded.

It remains to consider the case that $(n, q+1)$ is divisible by the full ℓ -part of $q+1$, so $\ell|n$. Here, for $n \geq 7$ we argue using maximal tori corresponding to cycle shapes $(n-2)(1)^2$, $(n-4)(3)(1)$ and $(n-4)(2)^2$, which by Lemma 4.1 all contain regular elements with non-central ℓ -part, to rule out all proper 2-split Levi subgroups as $C_{G^*}^\circ(s)$. The cases $n = 5, 6$ are excluded precisely as in the proof of Proposition 3.9. Finally, when $n = 3$ the only non-unipotent characters of degree not divisible by $\ell = 3$ are the three cuspidal characters of degree $(q-1)(q^2 - q + 1)/3$ when $q \equiv 2, 5 \pmod{9}$. These are indeed irreducible modulo 3 by [13, Lemma 4.3]. By [13, Thm. 7.6] for $S = \mathrm{U}_3(q)$ they are the characters of the Green correspondents of 1-dimensional modules of $N_S(P)$ for $P \in \mathrm{Syl}_\ell(S)$. Hence they are endotrivial by Theorem 2.2 as their values on non-trivial 3-elements are 1 (see [13, Lemma 3.2]). They are not endotrivial for $G = \mathrm{SU}_3(q)$ as the centre $Z(G) \cong C_3$ acts trivially. \square

Remark 4.4. For $S = \mathrm{U}_3(q)$, $\ell = 3$, $q \equiv 2, 5 \pmod{9}$, the group of torsion endotrivial modules $TT(S)$ identifies via Green correspondence with a subgroup of the group $X(N) \cong (\mathbb{Z}/2)^2$ of linear characters of $N := N_S(P)$ for $P \in \mathrm{Syl}_3(S)$. So in fact we have proved that S has three simple torsion endotrivial modules. Hence $TT(\mathrm{U}_3(q)) \cong (\mathbb{Z}/2)^2$, generated by the simple endotrivial modules V identified in Proposition 4.3

4.3. The general case.

Theorem 4.5. *Let $G = \mathrm{SU}_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$. Let $\ell \nmid q$ be such that the Sylow ℓ -subgroups of G are non-cyclic. Let V be a non-trivial simple kS -module for some central factor group S of G with k of characteristic ℓ . Then V is endotrivial if and only if $S = \mathrm{U}_3(q)$, $\ell = 3$, $q \equiv 2, 5 \pmod{9}$ and $\dim V = (q-1)(q^2 - q + 1)/3$.*

Proof. Let $G = \mathrm{SU}_n(q)$ and ℓ a prime divisor of $|G|$ with $d := d_\ell(-q)$. We may assume that ℓ is odd by [16, Thm. 6.7]. If $d = 1$ then the claim is contained in Proposition 4.3. So now assume that $d > 1$, and write $n = ad + r$ with $0 \leq r < d$. Since the Sylow ℓ -subgroups of G are non-cyclic we have $a \geq 2$, so $n \geq 2d$. Let $\chi \in \mathrm{Irr}(G)$ be the character of a simple endotrivial kS -module. Then χ lies in some Lusztig series $\mathcal{E}(G, s)$, where $s \in G^* = \mathrm{PGU}_n(q)$ is semisimple. Now G contains maximal tori T_1, T_2, T_3 of types $(n-d, d)$, $(n-r, r)$ and $(n-d-1, d, 1)$, all of which contain regular semisimple ℓ -singular elements by Lemma 4.1. As χ is endotrivial, it cannot vanish on these elements. But then by [16, Prop. 6.4] the centralizer $C_{G^*}(s)$ contains maximal tori of these three types. Using again Lemma 3.2 we see that $C_{G^*}(s) = G^*$, so $s = 1$ and χ is a unipotent character. But there are no non-trivial simple endotrivial unipotent characters by Proposition 4.2. \square

Remark 4.6. The three simple endotrivial cuspidal modules of $\mathrm{U}_3(q)$ for $\ell = 3$ of dimension $(q-1)(q^2 - q + 1)/3$ are in analogy with the two simple endotrivial cuspidal modules of

$L_2(q)$ for $\ell = 2$ of dimension $(q-1)/2$ (see [16, Prop. 3.8]), which also lie in the Lusztig series of a quasi-isolated ℓ -element.

Proof of Theorem 1.1. The assertion was already proved in [16, Thm. 1.2] for alternating groups, for sporadic groups, for exceptional groups of Lie type and more generally for all groups of Lie type in their defining characteristic. The case of classical groups of types B_n , C_n , D_n and 2D_n is treated in [17, ??]. So the only remaining cases are the linear and unitary groups, and exceptional covering groups. For these, the claim follows from Theorem 3.10, Proposition 3.11 and Theorem 4.5. \square

4.4. Cyclic blocks. Next we determine the number of cyclic blocks of $SU_n(q)$ containing simple endotrivial modules.

Proposition 4.7. *Let $G = SU_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$. Let $\ell \nmid q$ be a prime such that the Sylow ℓ -subgroups of G are cyclic and let $d := d_\ell(-q)$. Then $\ell > 2$, $d > 1$ and the number $\text{sb}(G)$ of ℓ -blocks containing simple endotrivial modules equals*

$$\text{sb}(G) = \begin{cases} \gcd(q+1, n) & \text{if } n = d, \\ 6 & \text{if } n = d+2 \text{ and } q = 2, \\ q+1 & \text{else.} \end{cases}$$

Proof. By Brauer–Suzuki G does not have cyclic Sylow 2-subgroups, hence $\ell > 2$. Secondly, ℓ does not divide $q+1$ as $n > 2$. Therefore $d > 1$. Also $n < 2d$.

Let $P \in \text{Syl}_\ell(G)$. As in the proof of Proposition 3.12, we have again, by [16, Lem. 3.2], $\text{sb}(G) = \frac{1}{e}|X(H)| = \frac{1}{e}|H/[H, H]|_\ell$, where e denotes the inertial index of the principal block B_0 of G and $H := N_G(P) = N_G(\langle u \rangle)$ for $u \in P$ an element of order ℓ .

First by [8, (3B)] we have that $e = d$. Indeed since B_0 is a unipotent block, its elementary divisor is the polynomial $X - 1$ and we get $e = e_{X-1} = d_\ell(-q)$.

Next we compute $|H/[H, H]|_\ell$. First we assume $n = d$. As above, consider the twisted Steinberg endomorphism $F : \text{GL}_d \rightarrow \text{GL}_d$ with $\text{GL}_d^F = \text{GU}_d(q)$. Let \mathbf{T} denote the maximal torus of diagonal matrices. Consider the d -cycle $w = (1 \ 2 \ \dots \ d)$ in the Weyl group $W = N_{\text{GL}_d}(\mathbf{T})/\mathbf{T} \cong \mathfrak{S}_d$. Applying [19, Prop. 25.3] yields

$$\mathbf{T}^{wF} = \{\text{diag}(t, t^{-q}, t^{(-q)^2}, \dots, t^{(-q)^{d-1}}) \in \mathbf{T} \mid t^{q^d - (-1)^d} = 1\}$$

so that $\mathbf{T}^{wF} \cong C_{q^d - (-1)^d}$ and $N_{\text{GU}_d(q)}(\mathbf{T})/\mathbf{T}^{wF} \cong C_{\mathfrak{S}_d(w)} \cong C_d$. Therefore we have $N_{\text{GU}_d(q)}(\mathbf{T}) = \mathbf{T}^{wF} \rtimes C_d \cong C_{q^d - (-1)^d} \rtimes C_d$, where C_d acts via F . Now since P is cyclic and is also a Sylow ℓ -subgroup of $\text{GU}_d(q)$, we have $N := N_{\text{GU}_d(q)}(P) = N_{\text{GU}_d(q)}(\mathbf{T})$. Taking the intersection with G yields $H = N \cap G \cong C_{(q^d - (-1)^d)/(q+1)} \rtimes C_d$ and if $z \in \text{GU}_n(q)$ is a generator for $C_{q^d - (-1)^d}$, then $H/[H, H] \cong \langle z^{q+1} \rangle / \langle z^{(q+1)(q+1)} \rangle \rtimes C_d$. Whence $|X(H)| = cd$ with $c = \gcd(q+1, (q^d - (-1)^d)/(q+1)) = \gcd(q+1, d)$, and we obtain $\text{sb}(G) = c$ in this case.

Now assume $n > d$. Since P is cyclic we may write $n = d + r$ with $1 \leq r < d$ and regard P as a Sylow ℓ -subgroup of $\text{GU}_d(q) \times \{1\} \leq \text{GU}_d(q) \times \text{GU}_r(q)$. Then

$$H = N_{\text{SU}_n(q)}(P) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in G \mid A \in N_{\text{GU}_d(q)}(P), B \in \text{GU}_r(q) \right\}.$$

Let $\theta : \text{GU}_d(q) \rightarrow \text{SU}_n(q)$ denote the injective homomorphism defined by $\theta(A) := \begin{pmatrix} A & 0 \\ 0 & B(A) \end{pmatrix}$ where $B(A)$ is the diagonal matrix $\text{diag}(\det(A)^{-1}, 1, \dots, 1) \in \text{GU}_r(q)$. It follows

that $H = S \rtimes \theta(N)$ where $S := \{1\} \times \mathrm{SU}_r(q) \leq \mathrm{GU}_d(q) \times \mathrm{GU}_r(q)$ and $N := N_{\mathrm{GU}_d(q)}(P)$. Therefore $H/[H, H] \cong (S/[S, S])_{\theta(N)} \times N/[N, N]$, where the subscript $\theta(N)$ means taking the cofixed points with respect to the action of $\theta(N)$. It follows from the case $n = d$ above that $N/[N, N] \cong \langle z \rangle / \langle z^{q+1} \rangle \times C_d \cong C_{q+1} \times C_d$. Moreover $(S/[S, S])_{\theta(N)}$ is trivial when $r = 1$ or when S is perfect, in which cases we obtain $\mathrm{sb}(G) = q + 1$.

The only cases with S not perfect are when $(r, q) \in \{(2, 2), (2, 3), (3, 2)\}$. If $(r, q) = (2, 2)$, then $S/[S, S] \cong C_2$ so that the action of $\theta(N)$ must be trivial. Hence $H/[H, H] \cong C_2 \times C_{q+1} \times C_d$ and $\mathrm{sb}(G) = 6$. Finally if $(r, q) = (2, 3)$, then $S/[S, S] \cong C_3$, and if $(r, q) = (3, 2)$, then $S/[S, S] \cong C_2 \times C_2$, but in both cases the cofixed points $(S/[S, S])_{\theta(N)}$ are trivial. Hence $H/[H, H] \cong C_{q+1} \times C_d$ and $\mathrm{sb}(G) = q + 1$ in these cases as well. \square

As no description of the Brauer trees of $\mathrm{SU}_n(q)$ similar to that of [20] for $\mathrm{SL}_n(q)$ is available we do not provide here a statement analogous to Corollary 3.13.

4.5. Zeroes of characters. As one application of our previous considerations we obtain the vanishing result stated in Theorem 1.3:

Proof of Theorem 1.3. Among exceptional covering groups, the only cases with ℓ -rank at least 3 for ℓ not the defining prime are the covering groups of $\mathrm{U}_6(2)$ with $\ell = 3$ for which the claim can be checked from the known character tables. Note that $\mathrm{U}_4(2)$ has two characters of degree 10 which do not vanish on 3-singular elements, but as $\mathrm{U}_4(2) \cong \mathrm{S}_4(3)$, this is covered by case (1).

Now let G be a central factor group of $\mathrm{SL}_n(q)$. We may assume that $n \geq 3$ since else Sylow ℓ -subgroups of G are cyclic. Let $\chi \in \mathrm{Irr}(G)$ be non-trivial. If χ is not unipotent, the claim follows from the proofs of Proposition 3.9 and Theorem 3.10. For unipotent characters the proof of Proposition 3.8 gives the result, except for the Steinberg character χ_{St} and for the case that $d = 1$, $(q - 1)/(n, q - 1)$ is prime to ℓ and $n \leq 6$, so $(n, \ell) \in \{(5, 5), (6, 3)\}$. For χ_{St} note that when G has ℓ -rank at least three, there exists an ℓ -element whose centralizer contains a non-trivial unipotent element, and χ_{St} vanishes on the product (see [6, Thm. 6.4.7]). For $\mathrm{SL}_5(q)$ with $\ell = 5$ dividing $q - 1$ exactly once, all unipotent characters except those labelled by $(3, 2)$ and by $(2^2, 1)$ vanish on suitable regular semisimple elements, and the character labelled by $(2^2, 1)$ vanishes on the product of a 5-singular semisimple element with centralizer $(q + 1)(q - 1)^2 A_1(q)/5$ with a regular unipotent element in its centralizer (see the Chevie-table [9]). This just leaves case (2). For $\mathrm{SL}_6(q)$ all unipotent characters vanish on some 3-singular element.

Finally, let G be a central factor group of $\mathrm{SU}_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$. Let $\chi \in \mathrm{Irr}(G)$ be non-trivial. If χ is not unipotent, the claim follows from the proof of Proposition 4.3 and Theorem 4.5. For unipotent characters by Proposition 4.2 we only need to discuss the Steinberg character, for which the claim follows as in the case of $\mathrm{SL}_n(q)$, and the possibility that $\ell | (n, q + 1)$ with $(n, \ell) \in \{(5, 5), (6, 3)\}$. As before, using [9] we see that the only possibility is the one listed in (3). \square

5. EXCEPTIONAL TYPE GROUPS

In this section we investigate simple endotrivial modules for exceptional groups of Lie type. We first discuss the unipotent characters.

Proposition 5.1. *Let $G = G(q)$ be a quasi-simple exceptional group of Lie type and $\ell \nmid q$ a prime for which the Sylow ℓ -subgroups of G are non-cyclic. If $\chi \in \text{Irr}(G)$ is the character of a non-trivial simple unipotent endotrivial kG -module, then $G = F_4(2)$, $\ell = 5$ and $\chi = F_4^{II}[1]$.*

Proof. The candidates for characters of simple unipotent endotrivial modules of exceptional groups of Lie type of rank at least 4 were determined in [16, Prop. 6.9]. They are given below (where $d = d_\ell(q)$ and the notation for unipotent characters is as in [6, §13]):

G	d	ℓ	χ	G	d	ℓ	χ
$F_4(q)$	4	5	$F_4^{II}[1]$	${}^2E_6(q)$	4	5	${}^2E_6[1]; \phi_{16,5}$
$E_6(q)$	4	5	$\phi_{80,7}; D_4, r$	$E_8(q)$	10	31	$\phi_{28,68}$
$E_6(q)$	6	19	$\phi_{6,25}$				

The characters $\phi_{80,7}$ of $E_6(q)$ and $\phi_{16,5}$ of ${}^2E_6(q)$ are not endotrivial by Corollary 2.4. The cuspidal unipotent character $F_4^{II}[1]$ of $F_4(q)$ is simple endotrivial in characteristic $\ell \nmid \Phi_4$ if and only if $q = 2$, so $\ell = 5$. Indeed, for the case $q = 2$, firstly the known ordinary and modular character tables [22] show that $F_4^{II}[1]$ remains irreducible modulo 5. Secondly, $F_4^{II}[1]$ is the character of a trivial source module as $F_4(2)$ has a subgroup $H \cong \text{O}_8^+(2) : \mathfrak{S}_3$ such that $F_4^{II}[1] = e_0 \cdot \text{Ind}_H^{F_4(2)}(1_a)$, where 1_a is the non-trivial linear character of H and e_0 is the principal block idempotent of $F_4(2)$, so endotriviality follows from Theorem 2.2. For $q > 2$, let $s \in F_4(q)$ be a semisimple element of order Φ_4 with centralizer of semisimple type $B_2(q)$. Such elements exist for all q (namely, there are $q^2/4$ such classes when q is even, and $(q^2 - 1)/4$ when q is odd). By Lusztig's character formula the value of $F_4^{II}[1]$ on such an element equals $q(q - 1)^2/2$, which is larger than 1 when $q > 2$.

We next claim that the unipotent character $\phi_{6,25}$ of $G = E_6(q)$ is reducible modulo primes ℓ dividing Φ_6 . For this we use that the decomposition matrix of the corresponding Iwahori–Hecke algebra H of type E_6 embeds into the decomposition matrix of the unipotent blocks of G (see e.g. [10, Thm. 4.1.14]). According to the decomposition matrix of H at Φ_6 in [10, Tab. 7.13], the character $\phi_{6,25}$ (which is called $6'_p$ there) occurs in the reduction of the characters $\phi_{20,2}$ and $\phi_{30,15}$ (denoted $20'_p, 30'_p$ respectively). Now assume that the unipotent character $\phi_{6,25}$ remains irreducible modulo ℓ . Then its Brauer character would have to be a constituent of the reduction of the unipotent character $\phi_{20,2}$. But the latter has smaller degree than $\phi_{6,25}$, so this is not possible. Exactly the same argument applies to the unipotent character $\phi_{28,68}$ of $E_8(q)$ modulo Φ_{10} , by using the decomposition matrix in [10, Tab. 7.15].

We next consider the cuspidal unipotent character ${}^2E_6[1]$ of $G = {}^2E_6(q)$. It lies in the 8-member family \mathcal{F} of unipotent characters attached to the largest 2-sided cell of the Weyl group of G . Let s be a regular semisimple element in a maximal torus of G of order $\Phi_1\Phi_2\Phi_3\Phi_4$, (so) of order divisible by Φ_4 . (The automizer of such a torus is isomorphic to the centralizer of the corresponding parametrizing element in the Weyl group, of type $D_5(a_1)$, hence cyclic of order 12. It is then easy to see that such regular elements exist for all q .) The argument given in the proof of Proposition 3.3 shows that the values on s of the principle series almost characters in that family \mathcal{F} are equal (up to sign) to the values of the corresponding characters of the Weyl group on the class $D_5(a_1)$. But from the character table of the Weyl group we see that all of these vanish. Since semisimple

classes are uniform, this implies that all unipotent characters in \mathcal{F} vanish on s . Thus, ${}^2E_6[1]$ cannot be endotrivial.

Virtually the same reasoning applies to the unipotent character D_4, r of $E_6(q)$. It also lies in the 8-member family, and by the calculation above it vanishes on regular semisimple elements of the maximal torus of order $\Phi_1\Phi_2\Phi_4\Phi_6$. Again, such regular elements of order divisible by Φ_4 exist for all q . \square

Theorem 5.2. *Let $G = G(q)$ be a quasi-simple exceptional group of Lie type of rank at least 4, $\ell \nmid q$ a prime dividing $|G|$, and $d = d_\ell(q)$. If $\chi \in \text{Irr}(G)$ is the character of a non-trivial simple endotrivial kG -module, then one of the following occurs:*

- (1) *The Sylow ℓ -subgroups of G are cyclic;*
- (2) *$(G, \ell, \chi) = (F_4(2), 5, F_4^{II}[1])$;*
- (3) *$G = E_6(q)$ with $\ell = 5 \mid (q^2 + 1)$ and χ is the semisimple character in the Lusztig series of a semisimple element with centralizer ${}^2A_3(q)\Phi_1\Phi_4$;*
- (4) *$G = {}^2E_6(q)$ with $\ell = 5 \mid (q^2 + 1)$ and χ is the semisimple character in the Lusztig series of a semisimple element with centralizer $A_3(q)\Phi_2\Phi_4$; or*
- (5) *$G = E_7(q)$ with $\ell \mid (q^2 + 1)$.*

Proof. Let G be quasi-simple of exceptional Lie type and of Lie rank at least 4. If a Sylow ℓ -subgroup of G has rank larger than 2, then there are no examples by [16, Thm. 6.11]. An easy check on the order formulas (see [19, Tab. 24.1], for example) shows that the cases with Sylow ℓ -subgroups of rank 2 are precisely the following: $d = 3, 4, 6$ for $F_4(q)$, $d = 4, 6$ for $E_6(q)$, $d = 3, 4$ for ${}^2E_6(q)$, $d = 4$ for $E_7(q)$, and $d = 5, 8, 10, 12$ for $E_8(q)$. By [16, Prop. 6.3] if χ is the character of a simple endotrivial kG -module then it must lie in a Lusztig series $\mathcal{E}(G, s)$ such that $s \in G^*$ centralizes a Sylow d -torus of G^* . If $s = 1$, then χ is unipotent by definition, and this case was discussed in Proposition 5.1 and leads to case (2). So $s \neq 1$. The character tables for the three groups $F_4(2)$, $E_6(2)$ and ${}^2E_6(2)$ are known and it can be checked directly that no further case apart from the ones listed in (3) and (4) arises there. So we now also assume that $q \neq 2$ for types F_4, E_6 and 2E_6 . We go through the various possibilities for (G, d) with $s \neq 1$.

TABLE 2. Maximal Sylow tori

G	d	$C_{G^*}(S)'$	Φ_e	possible $C_{G^*}(s)$
$F_4(q)$	3	$A_2(q)$	$\Phi_1\Phi_2$	$C, A_2(q)^2$
	4	$B_2(q)$	$\Phi_1\Phi_2$	$C, B_4(q)$
	6	${}^2A_2(q)$	$\Phi_1\Phi_2$	$C, {}^2A_2(q)^2$
$E_8(q)$	5	$A_4(q)$	Φ_3	$C, A_4(q)^2$
	8	$A_1(q^4), {}^2D_4(q)$	Φ_3, Φ_4	$D_8(q)$
	10	${}^2A_4(q)$	Φ_4, Φ_6	$C, {}^2A_4(q)^2$
	12	${}^3D_4(q), {}^2A_2(q^2)$	Φ_3, Φ_4	—

a) First assume that a Sylow d -torus T_d of G^* is a maximal torus, hence in particular self-centralizing in G . The cases are listed in Table 2. As pointed out above, any endotrivial character of G must thus lie in the Lusztig series of an element $s \in T_d$. In the table

we give one or two centralizers $C = C_{G^*}(S)$ of certain Φ_d -tori S of G^* of order Φ_d . By [16, Prop. 6.4], if $\chi \in \mathcal{E}(G, s)$ is endotrivial, then $C_{G^*}(s)$ must contain conjugates of all maximal tori of C containing regular semisimple elements. In particular, $|C_{G^*}(s)|$ must be divisible by Zsigmondy primes for the cyclotomic polynomials Φ_e as given in the fourth column of the table. It is easily seen that the only possible such centralizers of elements $1 \neq s \in T_d$ are the ones listed in the last column. In particular, there are no cases when $d = 12$ for $E_8(q)$.

We consider the remaining cases in turn. For $G = F_4(q)$ and $d = 3$, if s has centralizer $A_2(q)\Phi_3$ then $\mathcal{E}(G, s)$ contains three characters, all of which have degree $\chi(1) \equiv \pm 24 \pmod{\Phi_3}$. Thus they can possibly be endotrivial only for $\ell \in \{5, 23\}$. But neither of these two primes has $d_\ell(q) = 3$. When s has centralizer $A_2(q)^2$ and hence is isolated of order 3 and $q \equiv 1 \pmod{3}$, then the nine characters χ in $\mathcal{E}(G, s)$ satisfy $\chi(1) \equiv \pm 8 \pmod{\Phi_3}$, so necessarily $\ell = 7$. Now let t be a regular semisimple 7-singular element in the Sylow 3-torus T_3 of G , of order prime to 3. Let $\theta \in \text{Irr}(T_3)$ be such that (T_3, θ) lies in the geometric conjugacy class of s . Then s, t have coprime order, and the stabilizer of θ in $W(T_3)$ has index 2. Thus $\chi(t)$ is divisible by 2 for any $\chi \in \mathcal{E}(G, s)$ by Proposition 3.5, so χ cannot be endotrivial. The arguments in the case that $G = F_4(q)$ with $d = 6$ are quite analogous. Here we need to consider the characters in Lusztig series corresponding to elements s with centralizer ${}^2A_2(q)^2$, which can be discarded as in the previous case.

When $G = F_4(q)$ with $d = 4$, for s with centralizer $B_2(q)\Phi_4$ the four characters in $\mathcal{E}(G, s)$ of degree not divisible by Φ_4 have degrees $\chi(1) \equiv \pm 24 \pmod{\Phi_4}$, so only $\ell = 5$ is possible. Now all semisimple elements of $F_4(q)$ are real, as the Weyl group contains -1 , so the characters in the series $\mathcal{E}(G, s)$ are self-dual. But only the one with Jordan correspondent in $\mathcal{E}(B_2(q), 1)$ of degree $q\Phi_2^2/2$ satisfies the condition of Corollary 2.3. On the other hand, that character is reducible modulo $\ell = 5$, as can be seen from the decomposition matrix of the Hecke algebra of type B_2 modulo Φ_4 . So this does not lead to examples. For the involution s with centralizer $B_4(q)$, the elements in $\mathcal{E}(G, s)$ satisfy $\chi(1) \equiv \pm 3, \pm 6 \pmod{\Phi_4}$, so again necessarily $\ell = 5$. But those of degree congruent 1 modulo 5 are reducible modulo Φ_4 , as can again be seen from the decomposition matrix of the relevant Hecke algebra.

Now consider $G = E_8(q)$. When $d = 5$, for s with centralizer $A_4(q)\Phi_5$ all characters in $\mathcal{E}(G, s)$ of full Φ_5 -defect have $\chi(1) \equiv \pm 120 \pmod{\Phi_5}$, so we must have $\ell = 11$. Again, all characters in $\mathcal{E}(G, s)$ are self-dual since semisimple elements in $E_8(q)$ are real, and the only characters of degree congruent 1 modulo 11 are those with Jordan correspondent in $\mathcal{E}(A_4(q), 1)$ of degree $q\Phi_2\Phi_4$ and $q^6\Phi_2\Phi_4$. Both of these are reducible modulo Φ_5 as can be seen from the corresponding Hecke algebra. For the isolated element s of order 5 with centralizer $A_4(q)^2$ the corresponding characters in $\mathcal{E}(G, s)$ satisfy $\chi(1) \equiv \pm 24 \pmod{\Phi_5}$, so cannot be endotrivial for primes ℓ with $d_\ell(q) = 5$. When $d = 10$ the same line of argument leads to the characters in Lusztig series $\mathcal{E}(G, s)$ for s with centralizer ${}^2A_4(q)\Phi_{10}$ and $\ell = 11$. Here the two characters with Jordan correspondents labelled by the partitions $(4, 1)$ and $(2, 1^3)$ satisfy the degree conditions. But by Lemma 3.6 they both vanish on the product of an ℓ -element with a regular semisimple element in the maximal torus of ${}^2A_4(q)$ of order $q^4 - 1$.

Finally, if $G = E_8(q)$ and $d = 8$, then s is isolated with centralizer $D_8(q)$. The character degrees of $\chi \in \mathcal{E}(G, s)$ satisfy $\chi(1) \equiv \pm 3, \pm 6 \pmod{\Phi_8}$, but as $d_\ell(q) = 8$ we have $\ell \geq 17$, so certainly $\chi(1) \not\equiv \pm 1 \pmod{\ell}$.

TABLE 3. Rank 2 cases for regular d in exceptional groups

G	d	$ T $	$ T_1 $	e	possible $C_{G^*}(s)$
$E_6(q)$	4	$\Phi_1^2 \Phi_4^2$	$\Phi_1 \Phi_2 \Phi_4 \Phi_6$	6	${}^2A_3(q) \Phi_1 \Phi_4, D_5(q) \Phi_1$
	6	$\Phi_3 \Phi_6^2$	$\Phi_1 \Phi_2 \Phi_4 \Phi_6$	4	${}^2A_2(q) A_2(q^2)$
${}^2E_6(q)$	4	$\Phi_2^2 \Phi_4^2$	$\Phi_1 \Phi_2 \Phi_3 \Phi_4$	3	$A_3(q) \Phi_2 \Phi_4, {}^2D_5(q) \Phi_2$
	3	$\Phi_3^2 \Phi_6$	$\Phi_1 \Phi_2 \Phi_3 \Phi_4$	4	$A_2(q) A_2(q^2)$

b) Now consider the pairs (G, d) collected in Table 3. In all these cases, d is a Springer regular number for the Weyl group of G . In particular, the centralizer of a Sylow d -torus S_d of G^* is a maximal torus T of G^* , whose order is indicated in the table. Furthermore there exists a maximal torus T_1 in G containing a regular semisimple ℓ -singular element x of G , of order divisible by Φ_e , with e as in the 5th column. Thus, if χ is endotrivial, it cannot vanish on x . But then by [16, Prop. 6.4], χ must lie in a Lusztig series $\mathcal{E}(G, s)$ with $T_1^* \leq C_{G^*}(s)$. As pointed out above, s must also lie in the centralizer T of the Sylow d -torus S_d of G^* . It is now easily seen using [3, Tab. 3] that the only centralizers of elements $1 \neq s \in T$ with that property are as given in the 6th column of Table 3. We consider these possibilities in turn.

First assume that $G = E_6(q)$ with $d = 6$. Then all characters $\chi \in \mathcal{E}(G, s)$ with $C_{G^*}(s) = {}^2A_2(q) A_2(q^2)$ have $\chi(1) \equiv \pm 6 \pmod{\Phi_6}$, so necessarily $\ell = 7$. As s has order 3, we may use Proposition 3.5 with t regular of order $\Phi_4 \Phi_6/3$ to conclude that these χ are not endotrivial. The same reasoning applies when $G = {}^2E_6(q)$ and $d = 3$. For $G = E_6(q)$ and $d = 4$, all characters in $\mathcal{E}(G, s)$ with $C_{G^*}(s) = D_5(q) \Phi_1$ have degree congruent to ± 3 or ± 6 modulo Φ_4 , so here $\ell = 5$. But the order of s divides Φ_1 , and $5 \nmid \Phi_1$ as $d_\ell(q) = 4$, so we may apply again Proposition 3.5 with t regular of order $\Phi_4 \Phi_6 / \gcd(\Phi_1, \Phi_4 \Phi_6)$ to rule out this case. The same applies to $G = {}^2E_6(q)$ and $C_{G^*}(s) = {}^2D_5(q) \Phi_2$. When $G = E_6(q)$ with $d = 4$ and $C_{G^*}(s) = {}^2A_3(q) \Phi_1 \Phi_4$ then all characters in $\mathcal{E}(G, s)$ have degree congruent to 0 or ± 24 modulo Φ_4 , so once again $\ell = 5$. There are four characters in that Lusztig series of degree prime to ℓ , and the two of larger degree are reducible modulo ℓ , as follows from the decomposition matrix of the Hecke algebra of type 2A_3 modulo Φ_4 . The character of second smallest degree vanishes on a (necessarily regular) element of order $(q^3 + 1) \Phi_4$, so only the semisimple character in such a series remains, yielding case (3). Similarly, for $G = {}^2E_6(q)$ we need to discuss the Lusztig series for semisimple elements with centralizer $A_3(q) \Phi_2 \Phi_4$. Here, the decomposition matrix of the Hecke algebra of type A_3 shows that only the semisimple character is irreducible modulo ℓ , as in case (4).

c) The only remaining case is when $d = 4$ for $E_7(q)$, as in case (5) of the conclusion. \square

Remark 5.3. We note that the irreducible characters χ of degree 1543879701 of $E_6(2)$ and ψ of degree 707107401 of ${}^2E_6(2)$, respectively, corresponding to cases (3) and (4) of Theorem 5.2 are not the characters of simple endotrivial modules in characteristic 5.

Indeed, let G be one of $E_6(2)$ or ${}^2E_6(2)$. Both characters are self-dual, therefore if the corresponding modules were simple endotrivial, then their class in the group $T(G)$ of endotrivial modules would be a torsion element of order 2 (see Corollary 2.3 and its proof). Moreover such a module has to be the Green correspondent of a 1-dimensional $kN_G(P)$ -module of order 2, where $P \in \text{Syl}_5(G)$. In both cases the normalizer of a Sylow 5-subgroup has shape $5^2 : 4\mathfrak{S}_4$ so that $X(N_G(P)) \cong C_4$ has a unique element 1_a of order 2. Now $G = E_6(2)$ has a maximal subgroup $N_G(P) \leq H \cong F_4(2)$, so that χ would have to be the character of the kG -Green correspondent of the simple $kF_4(2)$ -module affording the character $F_4^{II}[1]$ of Proposition 5.1 (which is itself the kH -Green correspondent of 1_a). But it can be computed that χ does not occur as a constituent of the induction of $F_4^{II}[1]$ to G . Hence a contradiction.

A similar argument holds for ψ using a maximal subgroup of ${}^2E_6(2)$ isomorphic to Fi_{22} and the self-dual simple endotrivial module of dimension 1001 of the latter group in Corollary 2.9(c).

We can exclude some further instances of Theorem 5.2(3) and (4) as not belonging to simple endotrivial modules:

Lemma 5.4. *Let q be a prime power with $5|(q^2 + 1)$.*

- (a) *Let χ be the semisimple character of $E_6(q)$ in a Lusztig series parametrized by a 5-element with centralizer ${}^2A_3(q)\Phi_1\Phi_4$. Then χ is reducible modulo 5.*
- (b) *Assume $\gcd(q, 6) = 1$ and let χ be the semisimple character of ${}^2E_6(q)$ in a Lusztig series parametrized by a 5-element with centralizer $A_3(q)\Phi_2\Phi_4$. Then χ is reducible modulo 5.*

Proof. First consider $G = E_6(q)$. Under our assumption, χ lies in a unipotent 5-block of G . Since χ is a semisimple character, it is an explicitly known linear combination of Deligne–Lusztig characters. From this one can compute that its restriction to $5'$ -elements coincides with the restriction of the following linear combination of unipotent character in the principal block:

$$-\phi_{1,0} - \phi_{6,1} + \phi_{15,4} + \phi_{D_4:3} + \phi_{81,6} - \phi_{80,7} - \phi_{90,8} + \phi_{D_4:21} + \phi_{81,10}.$$

The decomposition matrix of the principal 5-block of G was computed in [7, Table 12]. In fact, the cited result gives the correct entries when $25|(q^2 + 1)$, and lower bounds in case $5|(q^2 + 1)$. From this, one sees that the above virtual character is a positive sum of irreducible Brauer characters

$$\psi_{D_4:3} + \psi_{D_4:21} + \psi_{81,10},$$

where we have labelled the Brauer characters by the corresponding ordinary unipotent characters using the triangular shape of the decomposition matrix. So χ is not irreducible modulo 5.

The previous arguments also apply to $G = {}^2E_6(q)$. Here, the restriction of χ to $5'$ -classes agrees with the one of

$$-\phi_{1,0} + \phi'_{2,4} - \phi'_{1,12} - \phi'_{8,3} + \phi'_{9,6} + {}^2E_6[1] + \phi'_{6,6} - \phi_{16,5} + \phi''_{9,6}.$$

By [7, Table 26], for $\gcd(q, 6) = 1$ this is the following positive linear combination of Brauer characters

$$(1 + c_1 + c_4){}^2E_6[1] + \psi'_{6,6} + \psi''_{9,6},$$

with integers $c_1, c_4 \geq 0$ and with the same labelling convention as above. We conclude as before. \square

On the other hand, observe that if χ lies in a Lusztig series as in the Lemma, but corresponding to a $5'$ -element, then it will remain irreducible modulo 5.

5.1. Exceptional covering groups. We conclude our investigations by the consideration of faithful modules for exceptional covering groups of exceptional groups of Lie type.

Proposition 5.5. *Let G be one of the exceptional covering groups $2.F_4(2)$, $2.^2E_6(2)$ or $6.^2E_6(2)$, and V be a faithful simple endotrivial kG -module. Then V occurs in Table 4 (with elementary abelian Sylow ℓ -subgroups of order ℓ^2), or in Table 5 (with cyclic Sylow ℓ -subgroups).*

Proof. In the case of cyclic Sylow ℓ -subgroups, we may conclude by using the criteria in [16, Thm. 3.7] and information on the Brauer trees (kindly provided by Frank Lübeck in the case of $\ell = 13$ when $G = 6.^2E_6(q)$).

The Sylow ℓ -subgroups of the groups in question are non-cyclic only for $\ell \leq 7$. The ordinary character tables are known for all of these groups and the usual criteria only leave very few cases for $2.F_4(2)$ and $2.^2E_6(2)$. Excluding those characters of $2.F_4(2)$ which are reducible modulo 5 by using the known 5-modular character table (see [22]), and Corollary 2.3 to exclude the 1 521 172 224-dimensional module for $2.^2E_6(2)$, only the four characters listed in Table 4 remain, where for $2.^2E_6(2)$ it is not known whether this character remains irreducible modulo 5.

The 22 100-dimensional \mathbb{F}_7G -module V for $G = 2.F_4(2)$ affording the character 22100_1 is endotrivial by Theorem 2.2, since V is the Green correspondent of a 1-dimensional module for $N_G(P)$ with $P \in \text{Syl}_7(G)$, hence a trivial source module. Indeed, let e be the block idempotent corresponding to the 7-block of G containing 22100_1 , then there is a linear character $1_a \in \text{Irr}(H)$, where $H \cong 2 \times ({}^3D_4(2) : 3)$ is a maximal subgroup of G containing $N_G(P)$, such that $e \cdot \text{Ind}_H^G(1_a) = 22100_1$.

The 5-modular reductions of the two characters $12376_1, 12376_2 \in \text{Irr}(2.F_4(2))$ are also trivial source modules, hence endotrivial by Theorem 2.2. Indeed, the group $G = 2.F_4(2)$ has two non-conjugate subgroups H_1, H_2 isomorphic to $\text{O}_8^+(2) \cdot \mathfrak{S}_3$, and if e' denotes the block idempotent corresponding to the faithful 5-block of G containing both 12376_1 and 12376_2 , then we have $e' \cdot \text{Ind}_{H_1}^G(1_{H_1}) = 12376_1$ and $e' \cdot \text{Ind}_{H_2}^G(1_{H_2}) = 12376_2$.

(As in §2 and §3 ordinary irreducible characters are denoted by their degrees, and the labelling of characters and blocks is that of the GAP character table libraries [22].) \square

TABLE 4. Candidates in exceptional covering groups of exceptional groups

G	P	$\dim(V)$	G	P	$\dim(V)$
$2.F_4(2)$	5^2	12376, 12376	$2.^2E_6(2)$	5^2	17736576*
	7^2	22100			

(*) this character is possibly reducible, or not endotrivial

TABLE 5. Simple endotrivial modules for exceptional covering groups of exceptional groups with cyclic Sylow subgroups

G	ℓ	$ X(H) $	$ X(H) /e$	$\dim V$
$2.F_4(2)$	13	24	2	2380, 2380
$2.F_4(2)$	17	16	2	52, 1146600
$2.^2E_6(2)$	11	20	4	2432
$2.^2E_6(2)$	11			537472
$6.^2E_6(2)$	11	60	12	90419328, 11145019392 ($2\times$)
$6.^2E_6(2)$	11			2606204160, 5877256320 ($2\times$)
$2.^2E_6(2)$	13	24	2	2432, 45696
$6.^2E_6(2)$	13	72	6	22619520, 6962288256 ($2\times$)
$2.^2E_6(2)$	17	16	2	2432, 1521172224
$6.^2E_6(2)$	17	48	6	494208, 1521172224 ($2\times$)
$2.^2E_6(2)$	19	18	2	45696, 22583328768
$6.^2E_6(2)$	19	54	6	494208, 33949238400 ($2\times$)

6. ON THE LOEWY LENGTH OF PRINCIPAL BLOCKS

In [12], Koshitani, Külshammer and Sambale investigate principal ℓ -block B_0 of finite groups of Loewy length 4. They show that a necessary condition to have Loewy length $LL(B_0) = 4$ is the existence of a character $\chi \in \text{Irr}(G)$ such that $\chi(x) = -1$ for all ℓ -singular elements $x \in G$ and $\chi(1) \equiv -1 \pmod{|G|_\ell}$. More precisely, in this situation the projective cover of the trivial module affords the character $1_G + \chi$, and the Heller translate $\Omega(k)$, which is an endotrivial module, affords χ and has composition length 2. In particular by Brauer reciprocity the column of the decomposition matrix corresponding to the trivial Brauer character has exactly two non-zero entries, so the first Cartan invariant c_{11} of the principal block equals 2. See [12, Prop. 4.6, Cor. 4.7].

Furthermore, if $\ell \geq 5$ a reduction to simple groups is proven. This follows from [12, Prop. 4.10] together with [12, Prop. 2.10]. It is also shown that Theorem 1.2 holds in characteristic $\ell = 2$ [12, Thm. 4.5], as well as in odd characteristic for the alternating groups [12, Thm. 3.10 together with Thm. 2.10], the sporadic groups [12, Thm. 4.11], and for groups of Lie type in their defining characteristic [12, Thm. 4.12]. Concerning groups of Lie type in cross characteristic our previous results show the following.

Corollary 6.1. *Let G be one of the simple groups $L_n(q)$ or $U_n(q)$ with $n \geq 3$, and $2 \neq \ell \nmid q$ be such that the Sylow ℓ -subgroups of G are non-cyclic. Then the principal ℓ -block B_0 of G does not have Loewy length 4.*

Proof. If $G = L_n(q)$ (resp. $G = U_n(q)$) with $n > 3$ then it follows from the proofs of Proposition 3.8, Proposition 3.9 and Theorem 3.10 (resp. Proposition 4.2, Proposition 4.3 and Theorem 4.5) that for every character $\chi \in \text{Irr}(G)$ either $\chi(1) \not\equiv -1 \pmod{|G|_\ell}$ or χ vanishes on some ℓ -singular element of G . The same holds if $G = L_3(q)$ unless $\ell = 3$, $q \equiv 4, 7 \pmod{9}$ and χ is the Steinberg character or one of the three characters of degree $\Phi_2\Phi_3/3$. But then by [14, Table 4] the reduction modulo 3 of χ is not a sum of two

irreducible Brauer characters. Similarly for $G = \mathrm{U}_3(q)$ with $q \neq 2$, all characters in $\mathrm{Irr}(B_0)$ are discarded by the proofs of Proposition 4.2, Proposition 4.3 and Theorem 4.5, except the Steinberg character when $\ell = 3$ and $q \equiv 2, 5 \pmod{9}$. But, by [13, Lemma 4.3], the latter reduces modulo 3 as a sum of 5 irreducible Brauer characters. Therefore in all cases $LL(B_0) \neq 4$ by [12, Cor. 4.7] (note that $\mathrm{U}_3(2)$ is solvable). \square

To deal with the simple groups of exceptional type we need the following observation:

Lemma 6.2. *Let G be a finite group with a BN-pair, and $T = B \cap N$. Let ℓ be a prime dividing $|T|$. Assume that N does not act transitively on the set I of non-trivial linear characters of T of order a power of ℓ . Then the 1-PIM of G is a sum of at least three characters.*

Proof. Assume that $\rho = 1_G + \chi$ is projective. Then so is its Harish-Chandra restriction ${}^*R_T^G(\rho)$ to T . In particular it contains the 1-PIM of T . Now the character of the 1-PIM of T contains the sum over all linear characters of T of ℓ -power order. Assume that N has at least three orbits on this set, with representatives $\psi_1 = 1_T, \psi_2, \psi_3$ say. Then the $R_T^G(\psi_i)$ are disjoint, so χ occurs in exactly one of them, say for $i = 2$. But then by reciprocity it is clear that ${}^*R_T^G(\rho)$ cannot contain ψ_3 , a contradiction. \square

Proposition 6.3. *Let G be a simple group of exceptional Lie type and $\ell > 2$ a prime for which the Sylow ℓ -subgroups of G are non-cyclic. Then the principal ℓ -block B_0 of G does not have Loewy length 4.*

Proof. According to [12, Prop. 4.12] we may assume that ℓ is not the defining prime for G . By [10, Thm. 4.1.14] the decomposition matrix of the Hecke algebra of G embeds into the decomposition matrix of G . Thus, if the decomposition matrix of the Hecke algebra has the property that its first column contains at least three non-zero entries, we must have $c_{11} \geq 3$ and so $LL(B_0) \neq 4$.

The tables in [10, p. 375–386] show that for types $F_4, E_6, {}^2E_6, E_7$ and E_8 this condition is satisfied for the root-of-unity specialization of the corresponding Hecke algebra H whenever $d = d_\ell(q) > 1$ and Sylow ℓ -subgroups are non-cyclic. Now the entries in the decomposition matrix for a finite field specialization of H will be at least as large as for the root-of-unity specialization, so our claim follows for all these groups as long as $d > 1$. Similarly, by [10, p. 373] this holds for type G_2 when $d \neq 1$, type 3D_4 when $d \neq 1, 3$, and for ${}^2F_4(q^2)$ when $\ell \nmid (q^2 - 1)$.

To treat $d = 1$, so $\ell \mid (q - 1)$, we use Lemma 6.2 with the natural BN-pair and thus with T the maximally split torus. It can be checked readily that in all cases, the Weyl group has at least two orbits on the set of non-trivial ℓ -elements of $\mathrm{Irr}(T)$, e.g., by order reasons. For example in $G_2(q)$ we have either $|\mathrm{Irr}(T)|_\ell - 1 = 8$ (if $\ell = 3$), or $|\mathrm{Irr}(T)|_\ell - 1 \geq 15$, and none of these can be an orbit length for the Weyl group of order $|W| = 12$. This argument also applies to $d = 3$ for ${}^3D_4(q)$. \square

Taken together with the results of [12] and of [17, Thm. 7.1], this gives Theorem 1.2.

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